Preference Structures

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ONLINE APPENDIX

The following theorem extends Theorem 5.3 by allowing possibly empty-valued choice correspondences.

Theorem B.1. *Let $(X, \mathfrak{X})$ be any choice environment, and C a choice correspondence on $\mathfrak{X}$. If C is rationalized by at least one preference structure on X, then*

$$\bigvee \mathbb{P}(C) = \succsim_C .$$

Proof of Theorem B.1.

Throughout the proof, we will denote $\mathbf{R}_C$ by $\mathbf{R}$ to simplify the notation. (That is, for any x and y in X, we have $\{x\} = C\{x,y\}$ iff $x\ \mathbf{R}^>\ y$, and $\{x,y\} = C\{x,y\}$ iff $x\ \mathbf{R}^=\ y$.) Consequently, $\succ_C\ \subseteq \mathbf{R}^>$ and $\sim_C\ \subseteq \mathbf{R}^=$. We will use these facts below as a matter of routine. Also, when $\succsim\ \in \mathbb{P}(C)$, we write $M(S) = \text{MAX}(S, \succsim)$ for any $S \in \mathfrak{X}$. First, we borrow the following results, each corresponds to Lemma A.5, Claim 1, and Claim 3, from the proof of Theorem 5.3. We note that the proofs of these results do not rely on the assumption that the choice correspondence C is nonempty-valued.

Lemma B.2. *For any finite $S \in \mathfrak{X}$, $\{x\} = C(S)$ implies $\{x\} = C\{x,y\}$ for every $y \in S$.*

Lemma B.3. $\mathbf{R}$ *is $\succsim_C$-transitive.*

Lemma B.4. *For any $\succsim\ \in \mathbb{P}(C)$, $\succ\ \subseteq\ \succ_C$ and $\sim\ \subseteq\ \sim_C$.*

Next, we extend Lemma A.6 and Claim 2 given in the proof of Theorem 5.3, the proofs of which rely on the assumption of nonempty valued choice correspondences, to the present setting.

Lemma B.5. *For any $S \in \mathfrak{X}$ and $x \in S$,*

$$x \succsim_C y \quad \textit{implies} \quad C(S \cup \{y\}) \cap S = C(S).$$

Proof. Take any $S \in \mathfrak{X}$ and any $\succsim\ \in \mathbb{P}(C)$. Suppose that $x \in S$ and $x \succsim_C y$. If $x \sim_C y$, then

$$z \in C(S) \quad \Leftrightarrow \quad z \in C(S \cup \{x\}) \quad \Leftrightarrow \quad z \in C(S \cup \{y\}) \quad \Leftrightarrow \quad z \in C(S \cup \{y\}) \cap S$$

for all $z \in S$, so we obtain $C(S) = C(S \cup \{y\}) \cap S$ at once. Assume $x \succ_C y$ in what follows. Then, $x\ \mathbf{R}^>\ y$, so $y \succsim x$ cannot hold (because $\mathbf{R}$ extends $\succsim$). Besides, if there is a $z \in S$ with $z \succ y$, then $C(S \cup \{y\}) \subseteq S$, so, by Lemma A.4, $C(S \cup \{y\}) = C(S)$, and we are done. It remains to consider the case where $(x, y) \in \text{Inc}(\succsim)$ and $y \in M(S \cup \{y\})$.

We show that, if $C(S \cup \{y\}) = \varnothing$, then $C(S) = \varnothing$ (and thus the claim of the lemma holds in this case). To prove this, assume $C(S \cup \{y\}) = \varnothing$. Since $y \in M(S \cup \{y\})$, by Proposition 4.1, there exists a y' in $M(S \cup \{y\})$ such that $y' \,\text{tran}(\mathbf{R}|_{M(S\cup\{y\})})^> y$. (Note that $y'\, \mathbf{R}^> y$ and $y' \in M(S)$.) Fix any $z \in M(S)$, and first suppose that $z \notin M(S \cup \{y\})$ and $z \,\text{tran}(\mathbf{R}|_{M(S)})\, y'$. Then, there exists a $k \in \mathbb{N}$ and $a_0, \ldots, a_k$ in $M(S)$ such that $y \succ z = a_0\, \mathbf{R} \cdots \mathbf{R}\, a_k = y'$. Let $\ell = \max\{i \in [k] : y \succ a_i\}$.

Then, we have $\ell < k$, for otherwise $y \succ a_k = y'$ and thus $y\,\mathbf{R}^{>}\,y'$, a contradiction. Moreover, by the construction of ℓ, $\{a_{\ell+1}, \ldots, a_k\} \subseteq M(S \cup \{y\})$ and $y \succ a_\ell\,\mathbf{R}\,a_{\ell+1}\,\mathbf{R} \cdots \mathbf{R}\,a_k = y'$. Then, by $\succsim$-transitivity of $\mathbf{R}$, $y\,\mathbf{R}\,a_{\ell+1}\,\mathbf{R} \cdots \mathbf{R}\,a_k = y'$, that is, $y\,\mathrm{tran}(\mathbf{R}|_{M(S\cup\{y\})})\,y'$, a contradiction. Thus, we have shown that $z \notin M(S \cup \{y\})$ implies $y'\,\mathrm{tran}(\mathbf{R}|_{M(S)})^{>}\,z$. Next, suppose that $z \in M(S \cup \{y\})$. As $C(S\cup\{y\}) = \varnothing$, there exists a $z' \in M(S\cup\{y\})$ with $z'\,\mathrm{tran}(\mathbf{R}|_{M(S\cup\{y\})})^{>}\,z$. Here, we may w.l.o.g. assume that $z'\,\mathrm{tran}(\mathbf{R}|_{M(S\cup\{y\})})\,y'$ (relabelling if necessary). By contradiction, let $z\,\mathrm{tran}(\mathbf{R}|_{M(S)})\,z'$. Then, there exist a $k \in \mathbb{N}$ and $a_0, \ldots, a_k \in M(S)$ such that $z = a_0\,\mathbf{R} \ldots \mathbf{R}\,a_k = z'$, where $y \succ a_i$ must hold for at least one $i \in [k]$ (as $z'\,\mathrm{tran}(\mathbf{R}|_{M(S\cup\{y\})})^{>}\,z$). Define $\ell = \max\{i \in [k] : y \succ a_i\}$. Then, $\ell < k$, for $\ell = k$ would imply $y \succ z'\,\mathrm{tran}(\mathbf{R}|_{M(S\cup\{y\})})\,y'$ and thus $y\,\mathrm{tran}(\mathbf{R}|_{M(S\cup\{y\})})\,y'$, a contradiction. But, then, we have $\{a_{\ell+1}, \ldots, a_k\} \subseteq M(S \cup \{y\})$ and $y \succ a_\ell\,\mathbf{R}\,a_{\ell+1}\,\mathbf{R} \cdots \mathbf{R}\,a_k = z'$. By $\succsim$-transitivity of $\mathbf{R}$, this implies that $y\,\mathbf{R}\,a_{\ell+1}\,\mathbf{R} \cdots \mathbf{R}\,a_k = z'$, that is, $y\,\mathrm{tran}(\mathbf{R}|_{M(S\cup\{y\})})\,z'\,\mathrm{tran}(\mathbf{R}|_{M(S\cup\{y\})})\,y'$, which again yields a contradiction. A conclusion: $z'\,\mathrm{tran}(\mathbf{R}|_{M(S)})^{>}\,z$. Hence, in all contingencies, we have proved that there exists a $z' \in M(S)$ with $z'\,\mathrm{tran}(\mathbf{R}|_{M(S)})^{>}\,z$ for an arbitrarily fixed $z \in M(S)$. In view of Proposition 4.1, this completes to verify that $C(S) = \varnothing$. For the rest of the proof, we shall thus assume that $C(S \cup \{y\}) \neq \varnothing$.

Since $x \succ_C y$ implies that y does not belong to $C(S \cup \{y\})$, we have

$$C(S \cup \{y\}) \subseteq M(S \cup \{y\}) \cap S \subseteq M(S).$$

Now, we claim that $C(S \cup \{y\})$ is an $\mathbf{R}$-highset in $M(S)$ To see this, take any $\succsim$-maximal z in S that does not belong to $C(S \cup \{y\})$. If $z \in M(S \cup \{y\})$, then we clearly have $C(S \cup \{y\})\ \mathbf{R}^{>}\ z$ (because $C(S\cup\{y\})$ is an $\mathbf{R}$-highset in $M(S\cup\{y\})$. If $z \notin M(S\cup\{y\})$, then, since $y \in M(S\cup\{y\})\backslash C(S\cup\{y\})$, we have $C(S\cup\{y\})\ \mathbf{R}^{>}\ y \succ z$, which implies $C(S\cup\{y\})\ \mathbf{R}^{>}\ z$ by Lemma A.3. Conclusion: $C(S\cup\{y\})$ is an $\mathbf{R}$-highset in $M(S)$. As $C(S \cup \{y\})$ is, obviously, an $\mathbf{R}$-cycle, it follows from Corollary 4.2 that $C(S \cup \{y\}) = C(S)$. ■

Lemma B.6. *$\succsim_C$ is transitive.*

Proof. The same proof for Claim 2 works, but we invoke Lemma B.5 instead of Lemma A.6. ■

We now turn to the proof of Theorem B.1. In what follows, we write $\succsim^* = \bigvee \mathbb{P}(C)$ for notational brevity. It follows from Theorem 5.2 that $\succsim^* \in \mathbb{P}(C)$ and hence we obtain $\succsim^* \subseteq \succsim_C$ by Lemma B.4. So, we only need to show that $\succsim_C \subseteq \succsim^*$. Toward a contradiction, suppose otherwise. In particular, in the first half of the proof, let us we assume that $\succ_C \nsubseteq \succ^*$, that is, that there exist $a, b \in X$ with $a \succ_C b$ while $a \succ^* b$ does not hold. (Obviously, $a \sim^* b$ does not hold either by Lemma B.4.) Define a binary relation $\trianglerighteq$ on X by

$$\trianglerighteq\ = \succsim^* \cup (a^{\uparrow} \times b^{\downarrow}),$$

where $a^{\uparrow} = \{x \in X : x \succsim^* a\}$ and $b^{\downarrow} = \{x \in X : b \succsim^* x\}$. Then, $\trianglerighteq$ is obviously a proper superrelation of $\succsim^*$ (as $a \trianglerighteq b$), and moreover it is straightforward to show that $\trianglerighteq$ is a preorder on X.

Claim 1. $\trianglerighteq$ extends $\succsim^*$.

Proof of Claim 1. Suppose that $x \succ^* y$, but $x \triangleright y$ does not hold. As $\succsim^* \subseteq \trianglerighteq$, it then follows that $y \trianglerighteq x$. By definition of $\trianglerighteq$, this implies that $y \succsim^* a$ and $b \succsim^* x$. But, then, $b \succsim^* x \succ^* y \succsim^* a$ and thus $b \succ^* a$. This is a contradiction to $a \succsim_C b$ by Lemma B.4. □

Claim 2. $\mathbf{R}$ extends $\trianglerighteq$.

Proof of Claim 2. First, suppose that $x \trianglerighteq y$. If $x \succsim^* y$, then we have $x\,\mathbf{R}\,y$ at once (as $\succsim^*$ extends $\mathbf{R}$). Otherwise, we have $x \succsim^* a \succ_C b \succsim^* y$, which implies $x \succsim_C a \succ_C b \succsim_C y$ by Lemma B.4 and thus $x \succ_C y$ by Lemma B.6. So, we obtain $x, \mathbf{R}^{>}\,y$ in this case. Next, suppose that $x \triangleright y$. If $x \succsim^* y$, then $x \sim^* y$ cannot hold (for otherwise $y \trianglerighteq x$), and therefore we have $x \succ^* y$ and $x\,\mathbf{R}^{>}\,y$. If $x \succsim^* y$ is false, then we have $x \succsim^* a \succ_C b \succsim^* y$, in which case we have already proved that $x\,\mathbf{R}^{>}\,y$ follow. □

Claim 3. $\mathbf{R}$ is $\trianglerighteq$-transitive.

Proof of Claim 3. Assume $x\,\mathbf{R}\,y \trianglerighteq z$ for some $x, y, z \in X$. If $y \succsim^* z$, then we have $x\,\mathbf{R}\,z$ at once by $\succsim^*$-transitivity of $\mathbf{R}$. Otherwise, we have $x\,\mathbf{R}\,y \succsim^* a \succsim_C b \succsim^* z$. Then, $x\,\mathbf{R}\,a \succsim_C b \succsim^* z$ by $\succsim^*$-transitivity of $\mathbf{R}$, which, in turn, implies $x\,\mathbf{R}\,b \succsim^* z$ by Lemma B.3. Applying $\succsim^*$-transitivity of $\mathbf{R}$ once again yields $x\,\mathbf{R}\,z$. An analogous argument shows that $x \trianglerighteq y\,\mathbf{R}\,z$ implies $x\,\mathbf{R}\,z$. □

Claim 4. $C(S) \subseteq \text{MAX}(S, \trianglerighteq)$ for any $S \in \mathfrak{X}$.

Proof of Claim 4. Take any $S \in \mathfrak{X}$ and x in $C(S)$, and suppose that $x \notin \text{MAX}(S, \trianglerighteq)$ by contradiction. Then, there exists a $y \in S$ with $y \triangleright x$. Note that $y \succsim^* x$ must be false, for, $y \succ^* x$ would imply $x \notin C(S)$, while $y \sim^* x$ would imply $x \trianglerighteq y$. So, it follows that

$$y \succsim^* a \succ_C b \succsim^* x. \tag{1}$$

Applying Lemma A.4 and Lemma B.5 along with (1) yields $C(S) = C(S \cup \{a\}) \cap S = C(S \cup \{a, b\}) \cap S$. But, then, $x \in C(S \cup \{a, b\})$, and thus $b \in C(S \cup \{a, b\})$ by Proposition 4.7. This is a contradiction as $a \succ_c b$. □

Claim 5. $(\trianglerighteq, \mathbf{R})$ rationalizes C.

Proof of Claim 5. Take any $S \in \mathfrak{X}$, and put $A = \text{MAX}(S, \trianglerighteq)$ and $B = \text{MAX}(S, \succsim^*)$. Then, we have $C(S) = \bigcirc(B, \mathbf{R})$, while we wish to show that $C(S) = \bigcirc(A, \mathbf{R})$. Moreover, it follows from Claim 1 and Claim 4 that $C(S) \subseteq A \subseteq B$. If $A = B$, there is nothing to prove. So, assume that $B \backslash A \neq \varnothing$.

First, suppose that $C(S) \neq \varnothing$. In this case, since $C(S)$ is obviously an $\mathbf{R}$-cycle, it is enough by Corollary 4.2 to verify that $C(S)$ is an $\mathbf{R}$-highset in A. But this follows at once because $C(S)$ is an $\mathbf{R}$-highset in B. (Indeed, if $x \in C(S)$ and $y \in A \backslash C(S)$, then $y \in B \backslash C(S)$, and hence $x\,\mathbf{R}^>\,y$.)

Next, consider the case where $C(S) = \varnothing$. The assumption that $B \backslash A \neq \varnothing$ asserts that there exist $x \in B$ and $y \in S$ with $y \triangleright x$. As $y \succsim^* x$ is false (for, $y \succ^* x$ would imply $x \notin B$, while $y \sim^* x$ would imply $x \trianglerighteq y$), we have condition (1). Then, by applying Lemma A.4 and Lemma B.5 on (1) as in the proof of Claim 4, we obtain $C(S \cup \{a, b\}) \cap S = C(S) = \varnothing$. By Proposition 4.7, this in turn implies that $a \notin C(S \cup \{a, b\})$ (as $y \notin C(S \cup \{a, b\})$). Also, $b \notin C(S \cup \{a, b\})$ as $a \succ_C b$. A conclusion: $C(S \cup \{a, b\}) = \varnothing$. In view of Proposition 4.1, hence, we have

$$\max(B', \text{tran}(\mathbf{R}|_{B'})) = \varnothing, \tag{2}$$

where $B' = \text{MAX}(S \cup \{a, b\}, \succsim^*)$. Take any $z \in A$. Obviously, $z \in B$ (as $A \subseteq B$). Moverover, $a \succ^* z$ is false, for otherwise $y \succ^* z$ by (1), and thus $z \notin B$, a contradiction. Also, $b \succ^* z$ is false, for otherwise $y \trianglerighteq b \succ^* z$ (observe that (1) implies $y \trianglerighteq b$) and thus $y \triangleright z$ (as $\trianglerighteq$ extends $\succsim^*$), which implies that $z \notin A$, a contradiction. So, we conclude that $z \in B'$, thus verifying that $A \subseteq B'$. Now, for any $z \in A$, (2) implies that there exist z' and z'' in B' such that $z''\,\text{tran}(\mathbf{R}|_{B'})^>\,z'\,\text{tran}(\mathbf{R}|_{B'})^>\,z$. We claim that either z' or z'' must belong to A. Suppose otherwise. Then, there exists a $w \in S$ such that $w \triangleright z'$. Since $w \succsim^* z'$ is false (for, $w \succ^* z'$ would imply $z' \notin B'$, while $w \sim^* z'$ would imply $z' \trianglerighteq w$), we have $w \succsim^* a \succ_C b \succsim^* z'$. As $z' \in B'$, $b \succ^* z'$ cannot hold, and thus it follows that $b \sim^* z'$. The analogous resoning applies to derive $b \sim^* z''$ from $z'' \in B' \backslash A$. But, then, $z' \sim^* z''$, and thus $z'\,\mathbf{R}^=\,z''$. This is a contradiction as $z''\,\text{tran}(\mathbf{R}|_{B'})^>\,z'$. Noting that $\text{tran}(\mathbf{R}|_{B'})$ is a preorder, we have shown that, for any $z \in A$, there exists a $z' \in A$ such that $z'\,\text{tran}(\mathbf{R}|_{B'})^>\,z$ and thus $z'\,\text{tran}(\mathbf{R}|_A)^>\,z$. (Indeed, as $A \subseteq B'$, $z\,\text{tran}(\mathbf{R}|_A)\,z'$ would imply $z\,\text{tran}(\mathbf{R}|_{B'})\,z'$.) Equivalently,

$$\max(A, \text{tran}(\mathbf{R}|_A)) = \varnothing.$$

In view of Proposition 4.1, we conclude that $C(S) = \bigcirc(A, \mathbf{R}) = \varnothing$, as desired. □

It follows from Claim 2 and Claim 3 that $(\trianglerighteq, \mathbf{R})$ is a preference structure on X. Then, by Claim 5, $\trianglerighteq \in \mathbb{P}(C)$, and thus $\trianglerighteq \subseteq \succsim^*$. But this is a contradiction as $\trianglerighteq$ is a proper superrelation of $\succsim^*$. This completes the first half of the proof of Theorem B.1. In the rest of the proof, let us derive a contradiction by assuming that $\succsim_C \not\subseteq \succsim^*$ and $\succ_C \subseteq \succ^*$. These assumptions imply that there exist $a', b' \in X$ with $a' \sim_C b'$ while $a' \succsim^* b'$ does not hold. (Observe that $b' \succsim^* a'$ may not hold either, for otherwise $b' \succ^* a'$ and $a' \sim_C b'$, a contradiction to Lemma B.4.) Define a binary relation $\trianglerighteq'$ on X by

$$\trianglerighteq' = \succsim^* \cup (a'^{\uparrow} \times b'^{\downarrow}) \cup (b'^{\uparrow} \times a'^{\downarrow}).$$

Again, it is obvious that $\trianglerighteq'$ is a proper superrelation of $\succsim^*$.

Claim 6. $\trianglerighteq'$ is a preorder.

Proof of Claim 6. It is enough to verify transitivity of $\trianglerighteq'$. Suppose that $x \trianglerighteq' y \trianglerighteq' z$. By symmetry, we consider the following exhaustive four cases: (i) $x \succsim^* y \succsim^* z$, (ii) $x \succsim^* y$ and $(y, z) \in (a'^{\uparrow} \times b'^{\downarrow})$, (iii) $(x, y) \in (a'^{\uparrow} \times b'^{\downarrow})$ and $(y, z) \in (b'^{\uparrow} \times a'^{\downarrow})$, and (iv) $(x, y) \in (a'^{\uparrow} \times b'^{\downarrow})$ and $(y, z) \in (a'^{\uparrow} \times b'^{\downarrow})$. In case (i), we have $x \succsim^* z$ and thus $x \trianglerighteq' z$ at once. In case (ii), we have $x \succsim^* y \succsim^* a'$ and $b' \succsim^* z$, and hence $(x, z) \in (a'^{\uparrow} \times b'^{\downarrow}) \subseteq \trianglerighteq$. In case (iii), $x \succsim^* a' \succsim^* z$, so $x \succsim^* z$ and $x \trianglerighteq' z$ follow. Case (iv) never arise, for otherwise $b' \succsim^* y \succsim^* a'$ and thus $b' \succsim^* a'$, a contradiction. □

Claim 7. $\mathbf{R}$ is $\trianglerighteq'$-transitive.

Proof of Claim 7. Suppose that $x \,\mathbf{R}\, y \trianglerighteq' z$. If $y \succsim^* z$, then $x \,\mathbf{R}\, z$ follows at once by $\succsim^*$-transitivity of $\mathbf{R}$. If $(y, z) \in (a'^{\uparrow} \times b'^{\downarrow})$, then $y \succsim^* a' \sim_C b' \succsim^* z$, which implies $y \succsim_C a' \sim_C b' \succsim_C z$ by Lemma B.4 and thus $y \succsim_C z$ by Lemma B.6. Then, $x \,\mathbf{R}\, z$ follows by Lemma B.3. By symmetry, $x \,\mathbf{R}\, y$ and $(y, z) \in (b'^{\uparrow} \times a'^{\downarrow})$ imply $x \,\mathbf{R}\, z$ as well. An analogous argument shows that $x \trianglerighteq' y \,\mathbf{R}\, z$ implies $x \,\mathbf{R}\, z$. □

Claim 8. $\mathbf{R}$ extends $\trianglerighteq'$. Moreover, $\triangleright' \subseteq \succ_C$.

Proof of Claim 8. Suppose that $x \trianglerighteq' y$. If $x \succsim^* y$, then we have $x \,\mathbf{R}\, y$ at once (as $\mathbf{R}$ extends $\succsim^*$). If $(x, y) \in (a'^{\uparrow} \times b'^{\downarrow})$, then $x \succsim_C y$ follows as shown in the proof of Claim 7, but this implies $x \,\mathbf{R}\, y$. Similarly, $x \,\mathbf{R}\, y$ follows when $(x, y) \in (b'^{\uparrow} \times a'^{\downarrow})$. Next, suppose that $x \triangleright' y$. If $x \succsim^* y$, then $x \sim^* y$ is false (for otherwise $y \trianglerighteq' x$), and thus we have $x \succ^* y$, so $x \succ_C y$ by Lemma B.4 and $x \,\mathbf{R}^{>}\, y$. If $(x, y) \in (a'^{\uparrow} \times b'^{\downarrow})$, then

$$x \succsim^* a' \sim_C b' \succsim^* y. \tag{3}$$

Here, in (3), at least one of $x \succsim^* a'$ or $b' \succsim^* y$ must hold strictly, for otherwise $y \sim^* b' \sim_C a' \sim^* x$ and thus $y \trianglerighteq' x$, a contradiction. Then, applying Lemma B.4 and Lemma B.6 on (3) obtains $x \succ_C y$ and thus $x \,\mathbf{R}^{>}\, y$ as we sought. An analogous proof shows that $x \triangleright' y$ and $(x, y) \in (b'^{\uparrow} \times a'^{\downarrow})$ imply $x \succ_C y$ and $x \,\mathbf{R}^{>}\, y$. This completes to prove that $\mathbf{R}$ extends $\trianglerighteq'$. In the latter half of this proof, we showed that $x \triangleright' y$ implies $x \succ_C y$ in all contingencies. Hence, $\triangleright' \subseteq \succ_C$. □

Claim 9. $\succ^* = \triangleright'$.

Proof of Claim 9. Take any $x, y \in X$ with $x \succ^* y$. Then, $x \trianglerighteq' y$ at once by definition of $\trianglerighteq'$. Suppose that $y \trianglerighteq' x$ to derive a contradiction.. Since $y \succsim^* x$ is false, we must have either $(y, x) \in (a'^{\uparrow} \times b'^{\downarrow})$ or $(y, x) \in (b'^{\uparrow} \times a'^{\downarrow})$. In the former case, we have $b' \succsim^* x \succ^* y \succsim^* a'$ and thus $b' \succ^* a'$, a contradiction. An analogous argument derives a contradiction, $a' \succ^* b'$, in the latter case as well. Hence, we conclude that $x \triangleright' y$, and $\succ^* \subseteq \triangleright'$. Conversely, if $\triangleright' \nsubseteq \succ^*$, then $\succ_C \nsubseteq \succ^*$ by Claim 8, which is a contradiction to the ongoing hypothesis. The proof is complete. □

It follows from Claims 6 through 8 that $(\trianglerighteq', \mathbf{R})$ is a preference structure on X. By Claim 9, we have $\mathrm{MAX}(S, \succsim^*) = \mathrm{MAX}(S, \trianglerighteq')$ for all $S \in \mathfrak{X}$, implying that $\trianglerighteq' \in \mathbb{P}(C)$. Hence, $\trianglerighteq' \subseteq \succsim^*$, for $\succsim^*$ is the largest preorder in $\mathbb{P}(C)$. A necessary contradiction is derived as $\trianglerighteq'$ is a proper superrelation of $\succsim^*$. This completes the proof of Theorem B.1. ■

The following result complements the remark given for Theorem 5.4. Recall that, where C is a choice correspondence on $\mathbf{k}(X)$, a binary relation $\trianglerighteq$ on X is defined by $x \trianglerighteq y$ iff there exist a $k \in \mathbb{N}$ and $z_1, \ldots, z_k$ in X such that (i) $y \,\mathbf{R}_C\, z_1 \,\mathbf{R}_C \cdots \mathbf{R}_C\, z_k \,\mathbf{R}_C\, x$, (ii) $y \in C\{y, z_1, \ldots, z_k\}$, (iii) there exists an $S \in \mathbf{k}(X)$ with $y \in S$ and $\{x, z_1, \ldots, z_k\} \subseteq C(S)$, and (iv) $y \notin C\{x, y, z_1, \ldots, z_k\}$.

Theorem B.6. *Let X be a topological space, and let C be the choice correspondence on $\mathbf{k}(X)$ rationalized by a continuous preference structure $(\succsim, \mathbf{R})$ on X. Then,*

$$\bigwedge \mathbb{P}(C) = \mathrm{tran}(\trianglerighteq) \cup \triangle_X.$$

Proof of Theorem B.6.

Throughout the proof, we denote $\mathbf{R} = \mathbf{R}_C$ and $\succsim_C^{\min} = \text{tran}(\trianglerighteq) \cup \triangle_X$ for notational brevity.

Claim 1. If $x \rhd y$, then $x \succ' y$ for any $\succsim' \in \mathbb{P}(C)$.

Proof of Claim 1. Let $x \rhd y$, so that the conditions (i) through (iv) of the definition of $\rhd$ hold. Take any $\succsim' \in \mathbb{P}(C)$. Then, (ii) implies that no z_k, $k = 1, \ldots, m$, $\succ$-domnates y, and (iii) implies that $\{x, z_1, \ldots, z_m\} \subseteq \text{MAX}(\{x, y, z_1, \ldots, z_m\}, \succsim)$. So, if $x \succ y$ is false, then $\text{MAX}(\{x, y, z_1, \ldots, z_k\}, \succsim) = \{x, y, z_1, \ldots, z_k\}$. But, then, (i) implies that y belongs to the top-cycle of this set, which contradicts with (iv). So, we conclude that $x \succ y$. □

Claim 2. For any $\succsim' \in \mathbb{P}(C)$, $\succsim'$ is an extension of $\succsim_C^{\min}$.

Proof of Claim 2. By Claim 1, $\rhd \subseteq \succ'$ and thus $\text{tran}(\rhd) \subseteq \succ'$ as $\succ'$ is transitive. This inclusion implies that $\text{tran}(\rhd)$ is an asymmetric and transitive binary relation and $\succsim_C^{\min}$ is a partial order with $\text{tran}(\rhd)$ as its strict part. Clearly, $\triangle_X \subseteq \sim'$, and hence $\succsim'$ extends $\succsim_C^{\min} = \text{tran}(\rhd) \cup \triangle_X$. □

Claim 3. $(\succsim_C^{\min}, \mathbf{R})$ is a preference structure on X.

Proof of Claim 3. Since $\succsim$ extends $\succsim_C^{\min}$ by Claim 2, while $\mathbf{R}$ extends $\succsim$, it readily follows that $\mathbf{R}$ extends $\succsim_C^{\min}$. If $x \succsim_C^{\min} y \,\mathbf{R}\, z$, then $x \succsim y \,\mathbf{R}\, z$, and thus $x \,\mathbf{R}\, z$. Similarly, $x \,\mathbf{R}\, y \succsim_C^{\min} z$ implies $x \,\mathbf{R}\, z$. Therefore, $\mathbf{R}$ is $\succsim_C^{\min}$-transitive. □

Claim 4. $(\succsim_C^{\min}, \mathbf{R})$ rationalizes C.

Proof of Claim 4. Take any $S \in \mathbf{k}(X)$. By Claim 2, we know

$$C(S) \subseteq \text{MAX}(S, \succsim) \subseteq \text{MAX}(S, \succsim_C^{\min}).$$

Since $C(S)$ is an $\mathbf{R}$-cycle by the representation, we only need to show that $C(S)$ is an $\mathbf{R}$-highset in $\text{MAX}(S, \succsim_C^{\min})$. By contradiction, suppose that there exist $z \in C(S)$ and $y \in \text{MAX}(S, \succsim_C^{\min}) \setminus C(S)$ such that

$$y \,\mathbf{R}\, z. \tag{4}$$

As $C(S)$ is an $\mathbf{R}$-highset in $\text{MAX}(S, \succsim)$, if $y \in \text{MAX}(S, \succsim)$, then $z \,\mathbf{R}^{>}\, y$, and we derive a contradiction to (4). So, suppose that $y \notin \text{MAX}(S, \succsim)$. Then, there exists an $x \in \text{MAX}(S, \succsim)$ such that $x \succ y$. If $x \notin C(S)$, then $z \,\mathbf{R}^{>}\, x \succ y$ and hence $z \,\mathbf{R}^{>}\, y$, a contradiction to (4). So, let $x \in C(S)$ for the rest of the proof. Since $z \in C(S)$, $x \in C(S)$ and $C(S)$ is an $\mathbf{R}$-cycle, there exists a finite sequence $z_1, \ldots, z_k$ in $C(S)$ such that

$$y \,\mathbf{R}\, z = z_1 \,\mathbf{R}\, z_2 \,\mathbf{R} \cdots \mathbf{R}\, z_k \,\mathbf{R}\, x. \tag{5}$$

We know that $x \succ y$. If there are z_ℓ such that $z_\ell \succ y$, then find the smallest ℓ^* with $z_{\ell^*} \succ y$, and relabel $x := z_{\ell^*}$ and consider the sequence $z_1, \ldots, z_{\ell^*-1}$. (As $y \,\mathbf{R}\, z_1$, we have $\ell^* > 1$.) By doing so, we can w.l.o.g. assume that $z_\ell \succ y$ holds for no ℓ. As y can $\succ$-dominates no z_ℓ either (since each z_ℓ is a member of $C(S)$), $\text{MAX}(\{y, z_1, \ldots, z_k\}, \succsim) = \{y, z_1, \ldots, z_k\}$, and hence it follows from (5) that

$$y \in C(\{y, z_1, \ldots, z_k\}). \tag{6}$$

Moreover, by construction,

$$\{x, z_1, \ldots, z_k\} \subseteq C(S) \quad \text{and} \quad y \in S. \tag{7}$$

Lastly, we have

$$y \notin C(\{x, y, z_1, \ldots, z_m\}) \tag{8}$$

as $x \succ y$. By comparing the conditions (5)-(8) with (i)-(iv) of the definition of $\rhd$, it follows that $x \rhd y$. But this is a contradiction since $y \in \text{MAX}(S, \succsim_C^{\min})$. The proof is complete. □

By Theorem 5.4, $\bigwedge \mathbb{P}(C) \in \mathbb{P}(C)$, which in turn implies that $\succsim_C^{\min} \subseteq \bigwedge \mathbb{P}(C)$ by Claim 2. Conversely, Claim 3 and Claim 4 imply that $\succsim_C^{\min} \in \mathbb{P}(C)$, and thus $\bigwedge \mathbb{P}(C) \subseteq \succsim_C^{\min}$ since $\bigwedge \mathbb{P}(C)$ is the smallest preorder in $\mathbb{P}(C)$. Hence, $\bigwedge \mathbb{P}(C) = \succsim_C^{\min}$, completing the proof of Theorem B.6. ■