Preference Structures*

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Abstract

We suggest using two binary relations to describe the preferences of a rational economic agent. The first of these is transitive and captures the comparisons that the decision maker is able to make easily, perhaps because it reflects the joint rankings of a committee of experts, or because it agrees with the preferences of all of her potential selves, or because these are those comparisons that just seem "obvious" to her. The second one arises from what we observe the agent choose in the context of pairwise choice problems. As such, it is assumed to be complete, but not necessarily transitive. Imposing two natural consistency conditions on these relations yields what we call a *preference structure*. It is shown that this model allows for phenomena like rational choice, indecisiveness, imperfect ability of discrimination, regret, and advise taking, among others. Our main goal is to study the choice behavior that arises from preference structures which we model by using the notion of top-cycles. We find that this leads to a rich theory of choice with a considerable explanatory power, and still with a surprising amount of predictive power. The two main issues we focus on here is the *existence* of choice, and the *recoverability* of one's (unobservable) core preference relation from choice data. We find that the choice theory developed here, while much more general, possesses existence and uniqueness properties that parallel those of the classical theory of rational choice.

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1 Introduction

The classical way of describing the preferences of an economic agent on a given set X of choice prospects is to use a binary relation on X. If, according to this relation, a prospect x is ranked higher than another prospect y, we understand that the agent prefers having x to having y. To get some mileage from the model, one typically imposes some properties on this binary relation, often corresponding to a form of rationality on the part of the agent. In the most standard scenario, of course, we posit that this relation is complete and transitive, and then describe the items that the agent finds choosable from a given feasible menu as those that maximize it.

This model is not only elegant, but it also possesses an admirable degree of predictive power. However, its explanatory power is known to be limited. The experimental demonstrations of nontransitivity of revealed preferences of individuals, for instance, go back to Tversky (1969), which is to cite but one reference from a rather large literature (cf. Loomes and Day (2010)). Besides, numerous explanations and models that accommodate nontransitivity of preferences are offered in the literature, including regret theory (Loomes and Sugden (1982)), nontransitive indifference and similarity (Luce (1956), Fishburn (1970), Beja and Gilboa (1992), and Rubinstein (1988)), and framing effects (Kahneman and Tversky (1979) and Salant and Rubinstein (2008)). Similarly, if we wish to model the occasional indecisiveness of an agent, then the completeness hypothesis has to be dropped (as in, say, models of multi-criteria decision making and/or Knightian uncertainty). Moreover, if the economic agent under consideration is, in fact, a group of individuals (such as a board of directors or a family), then positing completeness and transitivity at the outset is not at all warranted. After all, the two most standard binary relations that are relevant in this case are the Pareto ordering (which is incomplete) and the majority voting rule (which is nontransitive).

One reason why the classical model of preferences lacks in explanatory power is that this approach tacitly views that all pairwise choice problems are evaluated in the same way. Yet, it is simply unrealistic to presume that every choice problem is equally revealing. Depending on the context, some choices may be "easy," even "trivial," for an agent, while others may be "hard" enough that she may feel justifiably insecure about them. For instance, most agents would choose the sure lottery that pays them \$10 over the one that pays \$5 "easily," while they may find ranking two complicated lotteries "difficult." Or, when a committee of experts tell the agent, unanimously, that alternative x is better than y, the agent is likely to regard deciding between x and y an "easy" problem, but if some of the experts favor x and others y, the problem may well become "hard." Similarly, comparing two social policies would be easy for a social planner when there is unanimous agreement about these policies in the society, but the choice problem may become "difficult" if some people back one policy, and the others desire the alternate.

The upshot is that the choices of a decision maker across (subjectively) "hard" choice problems may fail the strict requirements of rationality, and hence reveal a preference relation that is not transitive. (This is a well-known viewpoint in the literature; see, among others, Mandler (2005) for a formal treatment of it, and Costa-Gomes et al. (2019) for empirical support.) By contrast, the choices across "easy" pairwise choice problems (whichever these may be for the agent) would presumably abide by transitivity. But, unless all such problems are "easy" for the agent, these yield an incomplete ordering of the choice prospects.

While this description is quite simple, and apparently realistic, it cannot be captured by the classical approach of modeling one's preferences by means of a binary relation. In this paper, we instead propose a broader, alternative approach of modeling the preferences of a rational decision maker. This approach is built on two binary relations on X. The first of these, denoted as $\succsim$, captures the rankings that the agent is perfectly comfortable with. As it is unlikely that a rational agent would exhibit a cyclical choice pattern across pairwise choice problems that she can "easily" solve, we assume $\succsim$ is reflexive and transitive, but not necessarily complete. The second binary relation, denoted as $\mathbf{R}$, arises from what we observe the agent choose in the context of all pairwise choice problems. (Thus $\succsim$ may not be observable, but $\mathbf{R}$ is.) As it is generated also by "hard" choice problems, we allow $\mathbf{R}$ be nontransitive, but, naturally, we assume it is complete. Put precisely, when $x \mathbf{R} y$, it is understood that an outside observer has seen the agent choose x over y from the menu $\{x, y\}$ at some point.

As $\succsim$ and $\mathbf{R}$ are meant to describe the preferences of a "rational" person, they must be consistent with each other. We thus assume at the outset that both the weak and strict parts of $\mathbf{R}$ extend those of $\succsim$, respectively. That is, if the agent is certain that x and y are perfect substitutes for her – this is captured by the core relation $\succsim$ declaring x and y indifferent – then the revealed preference $\mathbf{R}$ maintains that x and y are indeed indifferent. Similarly, if the agent thinks x is "obviously" strictly better than y – this is captured by $\succsim$ ranking x strictly above y – then $\mathbf{R}$ reveals precisely this.

Our interpretation suggests that the connection between $\succsim$ and $\mathbf{R}$ should be even tighter than this. To wit, suppose $x \mathbf{R} y$ and $y \succsim z$ for some alternatives x, y and z. Thus, the agent declares x superior to y (although she may not be completely confident in this judgement) while she is sure that y is better for her than z. It then seems reasonable that the agent would prefer x over z, albeit, she may be insecure about this decision (that is, $x \mathbf{R} z$ holds, but not necessarily $x \succsim z$). As the analogous reasoning applies also to the case where $x \succsim y$ and $y \mathbf{R} z$, it makes sense to require $\mathbf{R}$ be *transitive with respect to* $\succsim$, which means

$$x \mathbf{R} y \succsim z \quad \text{or} \quad x \succsim y \mathbf{R} z \quad \text{implies} \quad x \mathbf{R} z$$

for every x, y and z in X.

This property provides further discipline for the preference model at hand. Put precisely, this model, which we refer to as a *preference structure* on X, is a pair of binary relations $(\succsim, \mathbf{R})$ on X such that (i) $\succsim$ is reflexive and transitive, (ii) $\mathbf{R}$ is complete, (iii) $\mathbf{R}$ is an extension of $\succsim$ (in terms of both indifference and strict preference) and (iv) $\mathbf{R}$ is transitive with respect to $\succsim$. Evidently, this model reduces to the classical one when

$\succsim = \mathbf{R}$, but in general, it is much richer.[1,2]

In the body of the paper, we provide a large number of examples that highlight the coverage of preference structures. In addition to the classical rational choice model, among these are the models of incomplete preferences, preferences with imperfect ability of discrimination, regret preferences, and preferences completed by the recommendations of a consultant (Sections 3.2-3). We then prove that for any preference structure $(\succsim, \mathbf{R})$, there is a set of preorders such that $\succsim$ is realized as the intersection of this set and $\mathbf{R}$ as the union of it (Section 3.4). Thus, the first component of any preference structure is a unanimity ordering, while the second component of it is a rationalizable preference in the sense of Cherepanov et al. (2013).

Our main findings are reported in Sections 4 and 5. In Section 4, we consider how one may think of an agent making her choices on the basis of her preference structure. That is, we define the set $C(S)$ of all possible choices of an economic agent from a given feasible menu S by using a preference structure. In the classical case, this is done by setting $C(S)$ as the set of all maximum elements of S with respect to the preference relation of the agent. The situation is less clear cut in the context of an arbitrarily given preference structure $(\succsim, \mathbf{R})$. What readily follows from our interpretation is that the agent would never choose an alternative x from S if there is another alternative in S that strictly dominates x in terms of the "sure" ordering $\succsim$. Thus, $C(S)$ must be contained in $\mathbf{MAX}(S, \succsim)$, the set of all maximal elements in S with respect to $\succsim$. We then posit that the "choosable" alternatives in S should "maximize" $\mathbf{R}$ on $\mathbf{MAX}(S, \succsim)$. Unfortunately, as $\mathbf{R}$ need not be transitive, there is no *a priori* reason for the existence of such maxima, even when S contains only three alternatives. This is a problem familiar from social choice theory, and it is often addressed by using an alternative notion of optima, such as the top-cycle solution, the uncovered set, the Banks set, etc.. We adopt the first of these here, and set $C(S)$ as the top-cycle in $\mathbf{MAX}(S, \succsim)$ with respect to $\mathbf{R}$.[3] This generalizes the rational choice paradigm (because, when $\succsim = \mathbf{R}$, this specification makes $C(S)$ the set of all maxima relative to $\succsim$). In addition, it captures many interesting choice frameworks, among which are the models of rational choice with incomplete preferences, some satisficing models such as choice with constant thresholds, and certain types of sequentially rational choice procedures (Section 4.5). Yet, it is not meant to be a boundedly rational choice model; it is certainly not primed to capture phenomena like the attraction effect, limited attention, choice overload, status quo bias, etc.. It is, instead, a model that extends the coverage of the standard rational choice model at a foundational level, and as we shall see, one that retains a considerable amount

[1]Relaxing (iii) to $\succsim \subseteq \mathbf{R}$ here leads to what we call *weak* preference structures. While their interpretation is less appealing, such structures are also of interest. First, as we shall see, any weak preference structure can be turned into a proper preference structure without affecting the associated choice behavior. Second, there are quite a number of specific preference models in the literature that violate (iii), but nevertheless fit to the mold of a weak preference structure.

[2]While uncommon, describing preferences through two binary relations instead of one is not new; the way our model is situated in the existing literature is explained at the end of Section 3.1. At the outset, however, we should note that we are not aware of any work that develops a choice theory on the basis of such a preference model, which is the primary focus of our work.

[3]Put precisely, $C(S)$ is the smallest subset of $\mathbf{MAX}(S, \succsim)$ such that every element in this set is ranked strictly higher than every $\succsim$-maximal element outside this set with respect to $\mathbf{R}$.

of predictive power (Section 4.6).

There are three major theoretical queries that any choice theory that shoots for acting as a foundational model, has to respond to. These concern the issues of existence, uniqueness, and behavioral characterization. We discuss these in turn.

(I) *Existence of Choice.* A choice theory without having good existence properties is unlikely to be universally appealing. This is a primary concern for any preference model that allows for nontransitive binary choice patterns; indeed, it is one of the main reasons why nontransitive preferences are rarely used in economic models. In Section 4.3, we prove that the existence properties of the present choice model matches those of the rational choice model exactly. As our first main theorem, we show that under the usual compactness and continuity hypotheses, any choice correspondence that is rationalized by a preference structure is sure to be nonempty-valued.

(II) *Recoverability of the Underlying Preference Model.* This issue concerns the *uniqueness* of a preference model that underlies a given choice theory. The rational choice model is, trivially, on impeccable footing in this front; a rational choice correspondence can be rationalized by exactly one preference relation.[4] The situation is far more complicated in the case of the choice theory developed in this paper. Suppose C is the observed choice correspondence of a person, and assume that it is rationalized by some (unknown) preference structure $(\succsim, \mathbf{R})$. The question is if we can elicit the agent's preference structure from C, at least to an extent. This is a fundamental query, because the core preference relation of the agent is unobservable, but it is this part of the preference structure of the agent that matters most for welfare analysis (in the sense of, say, Bernheim and Rangel (2007, 2009)).

Section 5 is devoted entirely to this query. We first observe that the revealed preference part $\mathbf{R}$ of agent's preference structure is uniquely identified from her choice correspondence C; this parallels the situation in classical rational choice theory. Next, we prove that while there may well be a multitude of (incomplete) preference relations that, when coupled with $\mathbf{R}$, would rationalize C, the set of all such (core) preferences – we denote this set by $\mathbb{P}(C)$ – possesses an unexpected structure. Our second main result in this paper shows that any collection of such preferences can be combined into a single, more decisive, preference relation which, when coupled with $\mathbf{R}$, rationalizes C. Moreover, there is a *most decisive* preference structure in $\mathbb{P}(C)$. This relation – let us call it $\succsim_C$ – is the most complete preference relation such that $(\succsim_C, \mathbf{R})$ rationalizes C. It is of obvious interest, because it is the preorder that allows us make unambiguous welfare comparisons for the agent most frequently, and it is perfectly observable.

This is, however, a theoretical result; existence of a most complete element in $\mathbb{P}(C)$ is one thing, computing it by using C explicitly is another. Fortunately, there is a simple characterization of $\succsim_C$; this is the third main result of our paper (Section 5.1). It turns out that the strict part of this relation corresponds precisely to the Bernheim-Rangel criterion (that is, $x \succ_C y$ iff y is never chosen in a feasible set that contains x), while its symmetric part renders two alternatives indifferent iff these alternatives are behaviorally equivalent in the sense of Eliaz and Ok (2006) and Riberio and Riella (2017). In sum,

[4]This statement presumes that the choice domain (the set of all feasible sets) is sufficiently rich; as formalized later, we operate under this hypothesis thoughout the paper.

from any choice correspondence rationalizable by a preference structure we can explicitly elicit the preference structure (i) which rationalizes that correspondence; and (ii) whose "sure" preferences exhibit the least amount of indecisiveness compatible with that choice correspondence.[5]

(III) *Behavioral Axiomatic Characterization.* The overall approach we adopt in this paper is that of behavioral economics. We outline a theory of preferences and choice, demonstrate how this theory extends the coverage of the classical theory by means of a large set of examples, derive basic implications of this theory, and work out its existence and uniqueness properties. An alternative, complementary approach would be that of axiomatic decision theory. This would ask for the determination of the behavioral content of our choice theory in terms of a complete axiomatic system. Put precisely, this approach would require us identify a set of (behavioral) axioms for a choice correspondence C (each being weaker than the classical Weak Axiom of Revealed Preference) that are necessary and sufficient for C to be rationalizable by some preference structure. This would fully characterize the predictive content of the choice model, thereby providing one with methods of testing the model experimentally. We regard this approach essential, to be sure. However, given the already sizable length of the present paper, we cannot adopt it here. A complete characterization of our choice model in the tradition of revealed preference theory is, instead, provided in a separate, companion paper by Evren, Nishimura and Ok (2019).

All proofs that are omited in the main text are contained in the Appendix.

2 Nomenclature

As we deal with somewhat nonstandard preference relations in this paper, we introduce here some terminology that pertains to the general theory of binary relations on an arbitrarily given nonempty set X.

Binary Relations. By a **binary relation** on X, we mean any nonempty subset of $X \times X$. But, for any binary relation $\mathbf{R}$ on X, we often adopt the usual convention of writing $x \mathbf{R} y$ instead of $(x, y) \in \mathbf{R}$. For any nonempty $Y \subseteq X$, by $x \mathbf{R} Y$, we mean $x \mathbf{R} y$ for every $y \in Y$. Moreover, for any binary relations $\mathbf{R}$ and $\mathbf{S}$ on X, we simply write $x \mathbf{R} y \mathbf{S} z$ to mean $x \mathbf{R} y$ and $y \mathbf{S} z$, and so on. For any nonempty subset S of X, the **restriction** of $\mathbf{R}$ to S is defined as the binary relation on S given by

$$\mathbf{R}|_S := \mathbf{R} \cap (S \times S).$$

For any element x of X, the **upper set** of x with respect to $\mathbf{R}$ is defined as $x^{\uparrow,\mathbf{R}} := \{y \in X : y \mathbf{R} x\}$, and the **lower set** of x with respect to $\mathbf{R}$ is $x^{\downarrow,\mathbf{R}} := \{y \in X : x \mathbf{R} y\}$. When either $x \mathbf{R} y$ or $y \mathbf{R} x$, we say that x and y are **R-comparable**, and put

$$\mathrm{Inc}(\mathbf{R}) := \{(x, y) \in X \times X : x \text{ and } y \text{ are not } \mathbf{R}\text{-comparable}\}.$$

[5]Under standard compactness conditions on the choice domain, one can also determine the least decisive "sure" preferences of the agent from her choice behavior. This is explored in Section 5.2.

This set is symmetric, that is, $(x, y) \in \mathrm{Inc}(\mathbf{R})$ iff $(y, x) \in \mathrm{Inc}(\mathbf{R})$. If $\mathrm{Inc}(\mathbf{R}) = \varnothing$, we say that $\mathbf{R}$ is **complete** (or **total**).

The **asymmetric** (or **strict**) **part** of a binary relation $\mathbf{R}$ on X is defined as the binary relation $\mathbf{R}^{>}$ on X with $x\ \mathbf{R}^{>}\ y$ iff $x\ \mathbf{R}\ y$ and not $y\ \mathbf{R}\ x$, and the **symmetric part** of $\mathbf{R}$ is defined as $\mathbf{R}^{=} := \mathbf{R}\backslash\mathbf{R}^{>}$. The **composition** of two binary relations $\mathbf{R}$ and $\mathbf{S}$ on X is defined as $\mathbf{R} \circ \mathbf{S} := \{(x, y) \in X \times X : x\ \mathbf{R}\ z\ \mathbf{S}\ y$ for some $z \in X\}$. We say that $\mathbf{S}$ is a **subrelation** of $\mathbf{R}$, and that $\mathbf{R}$ is a **superrelation** of $\mathbf{S}$, if $\mathbf{S} \subseteq \mathbf{R}$.

We denote the diagonal of $X \times X$ by $\triangle_X$, that is, $\triangle_X := \{(x, x) : x \in X\}$. A binary relation $\mathbf{R}$ on X is said to be **reflexive** if $\triangle_X \subseteq \mathbf{R}$, **antisymmetric** if $\mathbf{R}^{=} \subseteq \triangle_X$, **transitive** if $\mathbf{R} \circ \mathbf{R} \subseteq \mathbf{R}$, and **quasitransitive** if $\mathbf{R}^{>}$ is transitive. If $\mathbf{R}$ is reflexive and transitive, we refer to it as a **preorder** on X. (Throughout the paper, generic preorders are denoted as $\succsim$ or $\trianglerighteq$, and the asymmetric parts of $\succsim$ and $\trianglerighteq$ are denoted as $\succ$ and $\vartriangleright$, respectively.) Finally, an antisymmetric preorder on X is said to be a **partial order** on X. If X is endowed with a prespecified partial order, we may refer to it as a **poset**.

The **transitive closure** of a binary relation $\mathbf{R}$ on X is the smallest transitive superrelation of $\mathbf{R}$; we denote this relation by $\mathrm{tran}(\mathbf{R})$. This relation always exists; we have $x\ \mathrm{tran}(\mathbf{R})\ y$ iff there exist a $k \in \mathbb{Z}_+$ and $x_0, ..., x_k \in X$ such that $x = x_0\ \mathbf{R}\ x_1\ \mathbf{R} \cdots \mathbf{R}\ x_k = y$. Obviously, $\mathrm{tran}(\mathbf{R})$ is a preorder on X, provided that $\mathbf{R}$ is reflexive.

Extension of Binary Relations. Let $\mathbf{R}$ be a binary relation on X. If $\mathbf{S}$ and $\mathbf{S}^{>}$ are subrelations of $\mathbf{R}$ and $\mathbf{R}^{>}$, respectively, we say that $\mathbf{R}$ is an **extension** of $\mathbf{S}$ (or that $\mathbf{R}$ *extends* $\mathbf{S}$). If $\mathbf{R}$ extends $\mathbf{S}$ and it is total, we refer to it as a **completion** of $\mathbf{S}$.

Transitivity with Respect to another Binary Relation. Our main focus in this paper is on reflexive, but not necessarily transitive, binary relations. A useful concept in the analysis of such binary relations is the notion of *transitivity with respect to a binary relation.* Put precisely, given any two binary relations $\mathbf{R}$ and $\mathbf{S}$ on X, we say that $\mathbf{R}$ is **S-transitive** if $\mathbf{R} \circ \mathbf{S} \subseteq \mathbf{R}$ and $\mathbf{S} \circ \mathbf{R} \subseteq \mathbf{R}$, which means that either $x\ \mathbf{R}\ y\ \mathbf{S}\ z$ or $x\ \mathbf{S}\ y\ \mathbf{R}\ z$ implies $x\ \mathbf{R}\ z$ for any $x, y, z \in X$. This notion generalizes the classical concept of transitivity, for, obviously, $\mathbf{R}$ is $\mathbf{R}$-transitive iff it is transitive.

The Transitive Core. Let $\mathbf{R}$ be a reflexive binary relation on X. By the **transitive core** of $\mathbf{R}$, we mean the largest subrelation $\mathbf{S}$ of $\mathbf{R}$ such that $\mathbf{R}$ is $\mathbf{S}$-transitive, and denote this subrelation as $\mathsf{T}(\mathbf{R})$. It is plain that $\mathbf{R}$ is transitive iff $\mathbf{R} = \mathsf{T}(\mathbf{R})$. A folk theorem of order theory says that $\mathsf{T}(\mathbf{R})$ exists, it is a preorder, and it satisfies: $x\ \mathsf{T}(\mathbf{R})\ y$ iff $x^{\uparrow,\mathbf{R}} \subseteq y^{\uparrow,\mathbf{R}}$ and $y^{\downarrow,\mathbf{R}} \subseteq x^{\downarrow,\mathbf{R}}$ (cf. Cerreia-Vioglio and Ok (2018).) In order theory, especially in the context of interval orders, $\mathsf{T}(\mathbf{R})$ is sometimes called the *trace* of $\mathbf{R}$ (cf. Doignon et al. (1986)). Here we instead adopt the terminology of Nishimura (2018) who has recently provided an axiomatic characterization of the operator T.

3 Preference Structures

3.1 Introduction

Let X be a nonempty set which we take as the collection of all mutually exclusive choice prospects for an economic agent (who may itself be a collection of individuals, such as a board of directors, congress, or a family). This agent is entirely confident in the preferential ranking of *some* of the alternatives in X. We model these rankings by means of a binary relation $\succsim$ on X. So, when $x \succsim y$ for some $x, y \in X$, we understand that the agent is "sure" that x is better than y for her. Of course, $\succsim$ is unobservable (because we do not know when an agent is "sure" about her preferential rankings), but our interpretation mandates $\succsim$ be (reflexive and) transitive: If x is surely better than y, and y is surely better than z, it makes sense that x will be deemed surely better than z. However, and this is where the present theory begins to deviate from the standard theory of rational decision-making, there is no need for $\succsim$ to be complete. The agent may well find the comparison of some alternatives "difficult," an entirely realistic phenomenon.[6]

Suppose the agent is unable to rank two alternatives x and y with respect to $\succsim$. When confronted with the problem of choosing between x and y, one will nevertheless observe her make a decision.[7] So, in the case of this choice problem, if she chooses x over y, we say that "x is revealed preferred to y," and if we have somehow witnessed that she choose x over y at some observation point, and y over x in some other, we say that "x is revealed indifferent to y." As such, we model *all* pairwise rankings of the individual, "easy" ones as well as the "hard" ones, by means of a binary relation $\mathbf{R}$ on X (which is observable). The very interpretation of $\mathbf{R}$ mandates it be complete. However, it is only natural that "hard choices" may not act transitively: If the agent has chosen x over y with great difficulty, and was also conflicted about her choice between y over z, but has nonetheless chosen y over z, then it may well be the case she choose z over x (again with difficulty). This not only rings true by daily introspection, but is verified by numerous experimental studies (on the nontransitivity of preferences). Moreover, if our economic agent consists of a set of individuals, then even the most standard methods of aggregating constituent preferences (such as majority voting) may result in the revelation of nontransitive rankings of the alternatives.

These considerations suggest that we model the "preferences" of an economic agent by means of an ordered pair $(\succsim, \mathbf{R})$ of binary relations on X such that $\succsim$ is a preorder and $\mathbf{R}$ is complete. Moreover, these relations should be consistent in the sense that $x \succsim y$ implies $x \mathbf{R} y$; this simply means that if x is "surely" at least as desirable as y for the agent, we would observe her choose x over y. Put succinctly, our interpretation of things very much suggests that $\mathbf{R}$ be a *superrelation* of $\succsim$.

As a matter of fact, it makes sense to ask $\mathbf{R}$ act in coherence with $\succsim$ in a way that goes beyond this property. Suppose our agent declares that $x \mathbf{R} y$ and $y \succsim z$ for some alternatives x, y and z. We interpret this as saying that the agent likes x better than y, even though she may well be somewhat insecure about this decision, while she

[6] For example, the agent may be employing a committee of experts to help her in her decision making, or she may be a social planner on behalf of a collection of individuals (each with her own preference relation). In either of these cases, $\succsim$ may correspond to the rankings of the alternatives according to the unanimity (Pareto) rule. When the unanimity ranking works, the comparisons are "easy," but of course, there may be many cases in which this ranking does not apply. (See Example 3.6 below.)

[7] In principle, the agent may "choose" not to make a choice, but this necessitates that at least some pairwise choice problems (those that do not include the option of not choosing) to be designated as unobservable situations. As formalized later, we abstract away from such contingencies here by tacitly allowing all pairwise choice situations within our framework.

prefers y over z in complete confidence. But then it stands to reason that the "obvious" superiority of y over z for this agent would entail that she would like x better than z, but, of course, it is possible that she may not be secure in this judgement either (that is, $x \mathbf{R} z$ holds, but not necessarily $x \succsim z$). Consequently, and since the same reasoning applies when $x \succsim y \mathbf{R} z$ as well, it makes good sense to require $\succsim$ and $\mathbf{R}$ to satisfy the following:

$$x \mathbf{R} y \succsim z \quad \text{or} \quad x \succsim y \mathbf{R} z \quad \text{implies} \quad x \mathbf{R} z$$

for all $x, y, z \in X$. Put succinctly, we posit $\mathbf{R}$ be $\succsim$-*transitive.* This property is not only reasonable, but it also brings some discipline to the model, and allows us to learn quite a bit about $\succsim$ (which is unobservable) from $\mathbf{R}$ (which is observable).

These considerations prompt the following:

Definition. An ordered pair $(\succsim, \mathbf{R})$ is a **weak preference structure** on a nonempty set X if $\succsim$ is a preorder on X and $\mathbf{R}$ is a $\succsim$-transitive and complete superrelation of $\succsim$ on X. In this context, we refer to $\succsim$ as the **core preference relation** of the structure, and to $\mathbf{R}$ as its **revealed preference relation**.

There are two (non-nested) special cases of the notion of a weak preference structure $(\succsim, \mathbf{R})$ that are of immediate interest. The first one of these strengthens the connection between the core and revealed preferences by requiring $\mathbf{R}$ be an *extension* of $\succsim$. This amounts to requiring $x \succ y$ imply $x \mathbf{R}^{>} y$ (in addition to $\succsim \subseteq \mathbf{R}$), that is, if x is "surely" strictly better than y for the agent, then she would never choose y over x.[8] The second one keeps the connection between $\succsim$ and $\mathbf{R}$ as is, but require $\mathbf{R}$ itself be transitive. This model embodies a lot of rationality within, but as we shall see, it still entails a choice theory distinct from the classical rational choice theory. Furthermore, many instances of this version of weak preference structures have already been considered in the literature.

Definition. A weak preference structure $(\succsim, \mathbf{R})$ on a nonempty set X is said to be a **preference structure** on X, provided that $\mathbf{R}$ is a completion of $\succsim$.[9] In turn, a (weak) preference structure is said to be a **transitive (weak) preference structure** on X, if $\mathbf{R}$ is transitive.[10]

Our immediate task is to highlight the relation between weak preference structures and some similar constructs found in decision theory, as well as providing several concrete examples. We will turn to developing a theory of choice based on preference structures in Section 4.

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[8]This requirement is only natural. For, suppose $\succsim \subseteq \mathbf{R}$ holds, but $\succ \subseteq \mathbf{R}^{>}$ fails. Then, even though it is "obvious" to the agent that x is strictly better than y (that is $x \succ y$), we may have $x \mathbf{R}^{=} y$ which means that the agent could choose y over x at some point. For instance, where x is \$1000 and y is \$0, and most agents would "surely" rank the former strictly over y, we would then allow the agent reveal herself to be indifferent between \$1000 and \$0. Similarly, when the economic agent is the coalition of, say, ten individuals, all ten of whom vote for x over y, without the extension requirement, the model would permit the coalition be declared indifferent between x and y.

[9]In what follows, when we wish to emphasize that a preference structure is not weak, we will refer to it as a *proper* preference structure.

[10]Put simply, a *transitive weak preference structure* on X is an ordered pair $(\succsim, \mathbf{R})$ where $\succsim$ and $\mathbf{R}$ are preorders on X such that $\succsim \subseteq \mathbf{R}$ and $\mathbf{R}$ is complete.

Relation to the Literature. Modeling individual preferences by means of two binary relations, one incomplete and the other complete, is not new in decision theory. Especially in the literature on decision making under uncertainty, this method is employed by a number of studies. For example, in the Anscombe-Aumann framework, Gilboa et al. (2010) have used two binary relations, the first being a preorder (à la Bewley (1986)) and the second a complete preorder (à la Gilboa and Schmeidler (1989)). In the jargon introduced above, this model is a weak transitive preference structure.

There are only few studies that employ two binary relations to model individual preferences in our general setup. Both Mandler (2005) and Danan (2008) suggest distinguishing between one's core preferences – Mandler refers to these as *psychological preferences*, and Danan as *cognitive preferences* – from her revealed preferences. The model of Mandler (2005), whose outcome space is restricted to be an open subset of $\mathbb{R}^n_+$, is, essentially, a special type of preference structure – see Example 3.3 below – but Mandler's emphasis is on sequential, nontransitive choice that is nevertheless consistent with the core preferences. By contrast, Danan's model is of the form $(\succsim, \mathbf{R})$, where $\succsim$ and $\mathbf{R}$ are binary relations on (a topological space) X such that both $\succsim$ and $\mathbf{R}$ are complete, and $\succ \subseteq \mathbf{R}^>$. Danan uses this model to suggest a method of understanding when an individual who has been observed to choose an alternative x over y is, in fact, indifferent between x over y. Notably, his model is a preference structure iff $\succsim = \mathbf{R}$, that is, in the intersection of our model and Danan's lies only the classical model of (complete and transitive) preferences.

The two papers that are most closely related to the present work are Giarlotta and Greco (2013) and Giarlotta and Watson (2018). Both of these papers work with a weak preference structure $(\succsim, \mathbf{R})$ on X. (The latter paper refers to such a structure as a *complete bi-preference.*) Giarlotta and Greco (2013) impose the following additional requirement on this model: For any two alternatives x and y, either $x \succsim y$ or $y\ \mathbf{R}\ x$. This model, which is called the *necessary and possible preference* on X, declares any two alternatives that are "hard" to compare as revealed indifferent. As such, it appears rather restrictive to serve as a general model of individual preferences (but it has been useful in multi-criteria decision analysis; see Giarlotta (2018) for a survey on this matter.) On the other hand, Giarlotta and Watson (2018) consider, instead, imposing the (mutually exclusive) requirements $\succ \subseteq \mathbf{R}^>$ or $\mathbf{R}^> \subseteq \succ$ on $(\succsim, \mathbf{R})$. In their jargon, the first requirement leads to *monotonic complete bi-preferences* and the second to *comonotonic complete bi-preferences*. While the former model is identical to what we define here as preference structures, Giarlotta and Watson (2018) instead focus on exploring the structure of the latter model which is, to use their words, "quite different from that of a monotonic bi-preference, being more related to decision analysis and operations research rather than choice theory."

The main focus of the present paper is explore how preference structures could be used to model the "choices" of an agent in a way that generalizes the classical rational choice theory, and several of its variants. We are not aware of any work that studies this issue.

3.2 Examples

Unless stated otherwise, X stands for an arbitrary nonempty set in the following examples. Our aim in this section is to demonstrate the breadth of the model of (weak) preference structures.

Example 3.1. Let $\succsim$ be a complete preorder on X. Then, $(\succsim, \succsim)$ is a transitive preference structure on X. (Every complete preference relation may thus be thought of as a preference structure.)

Example 3.2. Let $\mathbf{R}$ be a total binary relation on X. Then, $(\triangle_X, \mathbf{R})$ is a preference structure on X. (Every total binary relation may thus be thought of as a preference structure.)

Example 3.3. Let $\succsim$ be a preorder on X. If $\mathbf{R}$ stands for $\succsim \cup \mathrm{Inc}(\succsim)$, then $(\succsim, \mathbf{R})$ is a preference structure on X. (This is, essentially, the model Mandler (2005) has considered in the context of consumer choice.)

Example 3.4. (*Aggregation by Social Welfare Criteria*) Fix a positive integer n, and let u_i be a real map on X for each $i = 1, ..., n$. Define a preorder $\succsim$ by $x \succsim y$ iff $u_i(x) \geq u_i(y)$ for each $i = 1, ..., n$, and the binary relations $\mathbf{R}^1$, $\mathbf{R}^2$, and $\mathbf{R}^3$ on X by

$$x \, \mathbf{R}^1 \, y \quad \text{iff} \quad \sum_{i=1}^{n} u_i(x) \geq \sum_{i=1}^{n} u_i(y),$$

$$x \, \mathbf{R}^2 \, y \quad \text{iff} \quad \min_{i=1,...,n} u_i(x) \geq \min_{i=1,...,n} u_i(y),$$

and

$$x \, \mathbf{R}^3 \, y \quad \text{iff} \quad \max_{i=1,...,n} (u_i(x) - u_i(y)) \geq 0,$$

respectively. We may think of $\succsim$ here as a *Pareto ordering*, while $\mathbf{R}^1$ and $\mathbf{R}^2$ correspond to the *utilitarian* and *Rawlsian* social welfare criteria, respectively. By contrast, $\mathbf{R}^3$, which often is nontransitive, is best viewed as a *justifiable preference* (to borrow the jargon used by Lehrer and Teper (2011)). It is readily checked that $(\succsim, \mathbf{R}^i)$ is a weak preference structure on X for each $i = 1, 2, 3$. In fact, $(\succsim, \mathbf{R}^1)$ is a transitive preference structure, while $(\succsim, \mathbf{R}^2)$ is a transitive weak preference structure (but it need not be a preference structure). By contrast, $(\succsim, \mathbf{R}^3)$ need not be a preference structure, nor need it be transitive.

Example 3.5. (*Aggregation by Majority Voting*) Let $\mathcal{P}$ be a nonempty finite family of total preorders on X. Then, $\bigcap \mathcal{P}$ is the Pareto ordering induced by this collection. In turn, we define the (*majority voting*) binary relation $\mathcal{P}_{\mathrm{maj}}$ on X as

$$x \, \mathcal{P}_{\mathrm{maj}} \, y \quad \text{iff} \quad |\{\succsim \in \mathcal{P} : x \succ y\}| \geq |\{\succsim \in \mathcal{P} : y \succ x\}|$$

for every $x, y \in X$. Then, $(\bigcap \mathcal{P}, \mathcal{P}_{\mathrm{maj}})$ is a preference structure on X.

Example 3.6. (*Cautious Expected Utility Theory*) Let I be a compact interval, and X the collection of all Borel probability measures on I. Take any nonempty collections $\mathcal{U}$ and $\mathcal{V}$ of continuous and strictly increasing real maps on I, and consider the binary relation $\succsim_{\mathcal{U}}$ on X defined by

$$p \succsim_{\mathcal{U}} q \quad \text{iff} \quad \int_X u dp \geq \int_X u dq \text{ for every } u \in \mathcal{U}.$$

In the terminology of Dubra, Maccheroni and Ok (2004), therefore, $\succsim$ is a preorder on X that admits an expected multi-utility representation. Next, consider the binary relation $\mathbf{R}_{\mathcal{V}}$ on X defined by

$$p \; \mathbf{R}_{\mathcal{V}} \; q \quad \text{iff} \quad \inf_{u \in \mathcal{V}} \left(\int_X u dp \right) \geq \inf_{u \in \mathcal{V}} \left(\int_X u dq \right).$$

In the terminology of Cerreia-Vioglio, Dillenberger and Ortoleva (2015), $\mathbf{R}_{\mathcal{V}}$ is a complete preorder on X that admits a cautious expected utility representation. When these two representations use the same set of utility functions, they become consistent with each other. That is, $(\succsim_{\mathcal{U}}, \mathbf{R}_{\mathcal{U}})$ is a transitive weak preference structure on X. This need not be a preference structure, however.

Example 3.7. (*Preferences with Imperfect Discrimination*) Let $\mathbf{R}$ be a complete and quasitransitive binary relation on X. Then, $(\triangle_X \sqcup \mathbf{R}^{>}, \mathbf{R})$ is a preference structure on X. This model allows us to capture the utility model of imperfect discrimination which goes back to Armstrong (1939) and Luce (1956), and is studied more recently by Beja and Gilboa (1992), among others. To wit, let $u : X \to \mathbb{R}$ be any function and take any real number $\varepsilon \geq 0$. Define the binary relation $\mathbf{R}$ on X as $x \; \mathbf{R} \; y$ iff $u(x) \geq u(y) - \varepsilon$. This is a complete and quasitransitive binary relation on X with $x \; \mathbf{R}^{>} \; y$ iff $u(x) > u(y) + \varepsilon$ and $x \; \mathbf{R}^{=} \; y$ iff $|u(x) - u(y)| \leq \varepsilon$. (The idea is that the agent does not discriminate between alternatives whose utility values are close enough; Luce (1956) thus refers to ε as the *just noticeable difference*.) Then, $(\succsim, \mathbf{R})$ is a preference structure on X where $\succsim$ is the semiorder on X defined by $x \succsim y$ iff either $x = y$ or $u(x) > u(y) + \varepsilon$. The interpretation is that the pairwise ranking of any two alternatives is an "easy" one if the utilities of these alternatives are sufficiently (that is, more than ε) distinct, and "hard" otherwise.

Example 3.8. (*Preferences with Regret*) Let n be any positive integer, and $p := (p_1, ..., p_n)$ a probability vector with $p_i > 0$ for each i. Consider an environment in which there are n many states of the world, and state i obtains with probability p_i. We put $X := \mathbb{R}^n$, and interpret any $x := (x_1, ..., x_n) \in X$ as a state-contingent claim that pays x_i dollars at state i. Let $\succsim$ be the preorder defined by $x \succsim y$ iff $x_i \geq y_i$ for each $i = 1, ..., n$. (Thus, when $x \succsim y$ holds, x is an obviously better prospect than y.) Now let $u : \mathbb{R} \to \mathbb{R}$ and $Q : \mathbb{R} \to (-1, 1)$ be strictly increasing functions with $u(0) = 0$. Furthermore, assume that Q is odd (i.e., $Q(-a) = -Q(a)$ for every $a \in \mathbb{R}$), and convex on $\mathbb{R}_+$. We define the binary relation $\mathbf{R}$ on X as

$$x \; \mathbf{R} \; y \quad \text{iff} \quad \sum_{i=1}^{n} p_i Q\left(u(x_i) - u(y_i)\right) \geq 0.$$

This relation, due to Loomes and Sugden (1982), is known as a *regret preference*; it ranks prospects on the basis of their aggregate regret/rejoice due to the (utility) difference between the realized rewards. When $n \geq 3$ and Q is strictly convex on $\mathbb{R}_+$, $\mathbf{R}$ is not transitive, but it is always complete. In fact, it is easy to check that $(\succsim, \mathbf{R})$ is a preference structure on X.

The following example is a generalization of the previous one.

Example 3.9.[11] (*Intra-Dimensional Comparison Heuristics*) Let n be any positive integer, and consider an environment in which every commodity is modeled through n attributes. We thus put $X := \mathbb{R}^n$, and interpret any $x := (x_1, ..., x_n) \in X$ as a commodity which possesses x_i units of the attribute i. For each $i \in \{1, ..., n\}$, let us pick any skew-symmetric function $f_i : \mathbb{R}^2 \to (-1, 1)$ that is strictly increasing in the first component, and any strictly increasing and odd $W : (-1, 1)^n \to \mathbb{R}$.[12] We define the binary relation $\mathbf{R}$ on X as

$$x \mathbf{R} y \quad \text{iff} \quad W(f_1(x_1, y_1), ..., f_n(x_n, y_n)) \geq 0.$$

Here the vector $(f_1(x_1, y_1), ..., f_n(x_n, y_n))$ corresponds to comparisons of the goods x and y attribute by attribute; we can interpret f_i as measuring either the (dis)similarity of x_i and y_i or the salience of the ith attribute relative to the other attributes. We thus follow Tserenjigmid (2015), who has recently worked out a nice axiomatization for it, by calling $\mathbf{R}$ an *intra-dimensional comparison* (IDC) *relation*. Not only is any regret preference is an IDC relation, but the model of IDC relations contains the additive utility model (Example 3.5), the additive difference model of Tversky (1969) and a version of the salience theory of Bordalo, Gennaioli and Schleifer (2012). The upshot here is that $(\succsim, \mathbf{R})$ is a preference structure on X, where $\succsim$ is the binary relation on X defined by $x \succsim y$ iff $f_i(x_i, y_i) \geq 0$ for each $i = 1, ..., n$.[13]

Example 3.10. Let $\mathbf{R}$ be a complete binary relation on X, and recall that $\mathsf{T}(\mathbf{R})$ stands for the transitive core of $\mathbf{R}$ (Section 2). Then, $(\mathsf{T}(\mathbf{R}), \mathbf{R})$ is a weak preference structure on X, and, if $(\succsim, \mathbf{R})$ is a preference structure on X, then $\succsim$ must be a subrelation of $\mathsf{T}(\mathbf{R})$. But $(\mathsf{T}(\mathbf{R}), \mathbf{R})$ need not be a preference structure on X. For example, let u and ε be as in Example 3.7, let $\mathbf{R}$ be the semiorder defined in that example, and assume that $\sup u(X) - \inf u(X) > 2\varepsilon$. Then, as proved by Nishimura (2018), we have $x \mathsf{T}(\mathbf{R}) y$ iff $u(x) \geq u(y)$. So, $\mathsf{T}(\mathbf{R})^{>} \subseteq \mathbf{R}^{>}$ fails, and hence, $(\mathsf{T}(\mathbf{R}), \mathbf{R})$ is not a preference structure on X.

3.3 Weak vs. Proper Preference Structures

By definition, the strict part of the revealed preferences of a given preference structure extends the strict part of the core preference relation of that structure. A weak preference

[11] We thank Pietro Ortoleva for suggesting this example to us.

[12] *Skew-symmetry* of f_i means that $f_i(a, b) = -f_i(b, a)$ for every $a, b \in \mathbb{R}$.

[13] Due to the skew-symmetry and monotonicity of f_is, we actually have $x \succsim y$ iff $x_i \geq y_i$ for each $i = 1, ..., n$ here. This makes the proof of our claim routine.

structure may fail this property, thereby not qualifying to be a preference structure. (See Examples 3.4 and 3.6.) But the lack of this property is the only thing that separates a weak preference structure from a preference structure. That is, if $(\succsim, \mathbf{R})$ is a weak preference structure on an alternative set X, the only reason why this may not be a preference structure is that there may be alternatives x and y in X such that the agent inherently prefers x over y strictly (that is, $x \succ y$) and yet the revealed preference $\mathbf{R}$ views x and y equally desirable (that is, $x\ \mathbf{R}^=\ y$). If, therefore, we refine $\mathbf{R}$ so as to drop (y, x) from it (for any such x and y in X), we would obtain a preference structure. Put differently, there is a natural way of assigning a preference structure to any weak preference structure.

To formalize this discussion, let $(\succsim, \mathbf{R})$ be a weak preference structure on X. We define the binary relation $\mathbf{R}_{\succsim}$ on X as follows: $x\,\mathbf{R}_{\succsim}\,y$ iff

$$\text{either } x \succsim y \text{ or } [x \text{ and } y \text{ are not } \succsim\text{-comparable and } x\ \mathbf{R}\ y]. \tag{1}$$

In words, the ranking of any two alternatives by $\mathbf{R}_{\succsim}$ is done lexicographically. We first check if the core relation $\succsim$ applies, invoking $\mathbf{R}$ only when $\succsim$ is unable to rank the alternatives (which we interpret as when the agent have difficulties in comparing x and y). While this is not obvious, $(\succsim, \mathbf{R}_{\succsim})$ is indeed a preference structure on X.

Proposition 3.1. *Let $(\succsim, \mathbf{R})$ be a weak preference structure on a nonempty set X. Then, $(\succsim, \mathbf{R}_{\succsim})$ is a preference structure on X.*

The Natural Epimorphism. We can also look at the situation from a categorical point of view. Let $\mathrm{w}\mathcal{PS}_X$ stand for the set of all weak preference structures on X, and $\mathcal{PS}_X$ for the set of all preference structures on X. Then, the map $\pi : \mathrm{w}\mathcal{PS}_X \to \mathcal{PS}_X$, defined by $\pi((\succsim, \mathbf{R})) := (\succsim, \mathbf{R}_{\succsim})$, is a surjection that acts as the identity on $\mathcal{PS}_X$; we refer to π as the **natural epimorphism**. There is indeed good reason to consider this map as "natural." After all, we will see in Section 4 that $(\succsim, \mathbf{R})$ and $(\succsim, \mathbf{R}_{\succsim})$ are choice-theoretically equivalent. Put differently, according to the choice theory we build for preference structures in Section 4, there is no way of distinguishing between $(\succsim, \mathbf{R})$ and $(\succsim, \mathbf{R}_{\succsim})$. Simply put, the natural epimorphism π partitions $\mathrm{w}\mathcal{PS}_X$ in such a way that any one cell of the partition contains exactly one preference structure (which can be used as the representative of that cell).

A transitive weak preference structure may well fail to be a preference structure (Examples 3.4 and 3.6). Interestingly, the image of such a structure under the natural epimorphism, which, by Proposition 3.1, is a preference structure, may lose its transitivity. However, again in terms of the choice theory that we will introduce in Section 4, this does not make a difference. This is not really surprising, for the revealed preference part of such an image is sure to be quasitransitive.

Corollary 3.2. *Let $(\succsim, \mathbf{R})$ be a transitive weak preference structure on a nonempty set X. Then, $(\succsim, \mathbf{R}_{\succsim})$ is a preference structure on X and $\mathbf{R}_{\succsim}$ is quasitransitive.*

Modeling Choice by Consultation. There is a nice interpretation of the image of a transitive weak preference structure under the natural epimorphism. Think of an individual who, when faced with a "hard" choice problem, seeks the advise of a consultant.

The issue of dealing with "easy" choices is modeled by means of her (core) preference relation $\succsim$ on the given alternative space X. When two alternatives x and y are incomparable with respect to $\succsim$ – this choice is "hard" for the agent – she acts according to the advice of another individual (consultant). We imagine that the consultant is rational in the traditional sense, so her advice stems from a complete preorder $\mathbf{R}$ on X. Moreover, we assume that this preorder is consistent with $\succsim$ in the sense that $\succsim \subseteq \mathbf{R}$. (Otherwise, it would be unrealistic to presume that the agent trust the recommendations of the advisor, as some of those would conflict with her core preferences.) As such, $(\succsim, \mathbf{R})$ is a transitive weak preference structure on X. But, in this interpretation, $\mathbf{R}$ does not really correspond to the revealed preferences of the subject agent. After all, these preferences reflect those of the consultant only over the problems that this agent finds "hard," and thus seeks help for. In other words, the revealed preferences of the agent should coincide with $\succsim$ whenever $\succsim$ is able to render a ranking, and with $\mathbf{R}$ when this is not possible. Thus, the preference structure $\pi((\succsim, \mathbf{R}))$ seems like the "right" model that corresponds to this interpretation. In this model, the revealed preferences of the agent need not be transitive, but they are quasitransitive.

3.4 Characterization of Preference Structures

The following result, in which X is an arbitrary nonempty set, provides a general representation theorem for preference structures. Its import stems from the fact that it connects the two components of a preference structure by means of a single entity, namely, a collection of preorders.

Theorem 3.3. *Let $\succsim$ and $\mathbf{R}$ be binary relations on X. Then, $(\succsim, \mathbf{R})$ is a (weak) preference structure on X if, and only if, there is a nonempty collection $\mathcal{P}$ of preorders on X such that*

$$(\succsim, \mathbf{R}) = \left(\bigcap \mathcal{P}, \bigcup \mathcal{P}\right) \tag{2}$$

where $\bigcup \mathcal{P}$ is complete and each $\trianglerighteq \in \mathcal{P}$ extends (includes) $\succsim$.[14]

The "if' part of this result provides a general method of defining preference structures. In turn, its "only if" part provides a *multi-selves* interpretation for any given preference structure $(\succsim, \mathbf{R})$. To wit, let $\mathcal{P}$ stand for a nonempty collection of preorders on X as found in Theorem 3.3. We may think of each element $\trianglerighteq$ in $\mathcal{P}$ as a (potentially incomplete) preference relation of a different "self" of the same individual. (For instance, the agent may not know which of these relations will be the relevant one at the time of consumption, so entertains them all before making her choice.) These "selves" of the agent are consistent with the core preference relation $\succsim$ of the agent in that every one of them extends $\succsim$. In addition, $\succsim$, being equal to $\bigcap \mathcal{P}$, ranks an alternative x over another alternative y iff every one of her "selves" agrees that this is the correct ranking; $\succsim$ may thus be thought of as a *dominance* relation. On the other hand, the revealed preference relation $\mathbf{R}$ of the agent, being equal to $\bigcup \mathcal{P}$, ranks x over y iff at least one of her "selves" agrees that this is the correct ranking. In this sense, we may think of $\mathbf{R}$ as a

[14] A similar result for necessary and possible preferences was given by Giarlotta and Greco (2013).

rationalizable preference on X, borrowing (and slightly abusing) the terminology used by Cherepanov, Feddersen and Sandroni (2013). Importantly, these notions of dominance and rationalizability are compatible, for they are based on the preferences of the same set of "selves" of the agent.

Remark. A natural question is when we can guarantee the completeness of each member of $\mathcal{P}$ in the representation provided in Theorem 3.3. It turns out that this is a very restrictive requirement; we can do this only when $\mathbf{R}$ is obtained from $\succsim$ by rendering every $\succsim$-incomparable pair indifferent. Put more precisely: A preference structure $(\succsim, \mathbf{R})$ on X satisfies $\mathbf{R} = \succsim \sqcup \mathrm{Inc}(\succsim)$ iff there is a nonempty collection $\mathcal{P}$ of complete preorders on X such that (i) $(\succsim, \mathbf{R}) = (\bigcap \mathcal{P}, \bigcup \mathcal{P})$ and (ii) each $\trianglerighteq \in \mathcal{P}$ extends $\succsim$.[15] (We omit the proof, which is available upon request.)

As a final remark, we note that Theorem 3.3 modifies readily to give a characterization of transitive preference structures. This reads particularly simple if we impose the completeness assumption (for the second relation) at the outset:

Corollary 3.4. *Let $\succsim$ and $\mathbf{R}$ be binary relations on X with $\mathbf{R}$ being complete. Then, $(\succsim, \mathbf{R})$ is a transitive (weak) preference structure on X if, and only if, there is a nonempty collection $\mathcal{P}$ of preorders on X such that (2) holds, $\mathbf{R} \in \mathcal{P}$, and each $\trianglerighteq \in \mathcal{P}$ extends (includes) $\succsim$.*

4 Choice by Preference Structures

In Section 3 we have looked at various examples of preference structures, and their basic properties. In this section, we turn to how "choices" may arise from preference structures. This necessitates that we agree on what it means for an alternative to "maximize" a given complete (but not necessarily transitive) binary relation on a given feasible set, so we start the section with the discussion of this issue.

4.1 Maximization of Complete Binary Relations

Let X be a nonempty set, $\mathbf{R}$ a binary relation on X, and S a nonempty subset of X. An element x of S is called **R-maximal** in S if there is no $y \in S$ with $y \ \mathbf{R}^{>} \ x$, and **R-maximum** in S if $x \ \mathbf{R} \ S$. We denote the set of all $\mathbf{R}$-maximal and $\mathbf{R}$-maximum elements in S by $\mathbf{MAX}(S, \mathbf{R})$ and $\max(S, \mathbf{R})$, respectively. We always have $\max(S, \mathbf{R}) \subseteq \mathbf{MAX}(S, \mathbf{R})$, but this inequality may hold strictly (unless $\mathbf{R}$ is complete).

For a complete, but nontransitive, binary relation $\mathbf{R}$, these notions are rarely useful, because in this case $\mathbf{MAX}(S, \mathbf{R})$ may be empty even for a finite set S. For this reason, alternative notions of extrema are developed for binary relations. The best-known of these is the notion of *top-cycles* to which we now turn.

[15] A classical result of order theory, due to Dushnik and Miller (1941), says that every partial order $\succsim$ on a nonempty set X is the intersection of a nonempty collection $\mathcal{P}$ of linear orders on X. More generally, Donaldson and Weymark (1998) prove that every preorder $\succsim$ on X is the intersection of a nonempty collection $\mathcal{P}$ of total preorders on X. The theorem we just stated, which is based on the Axiom of Choice, not only generalizes the Donaldson-Weymark Theorem, but it also shows that all members of $\mathcal{P}$ in that theorem can be chosen as extensions of $\succsim$.

Top-Cycles. Let $\mathbf{R}$ be a complete binary relation on X. We say that a nonempty subset A of S is a **highset in S with respect to $\mathbf{R}$**, or more simply, an **$\mathbf{R}$-highset in S**, if

$$x \; \mathbf{R}^{>} \; y \quad \text{for every } x \in A \text{ and } y \in S \backslash A.$$

Notice that the collection of all $\mathbf{R}$-highsets in S is nonempty, because it contains S. Moreover, this collection is linearly ordered by set inclusion $\supseteq$.[16] Consequently, if it exists, there is a unique smallest $\mathbf{R}$-highset in S, namely, the intersection of all $\mathbf{R}$-highsets in S. We thus define the **top-cycle in S with respect to $\mathbf{R}$** as

$$\bigcirc(S, \mathbf{R}) := \bigcap \{A : A \text{ is an } \mathbf{R}\text{-highset in } S\}.$$

This set is nonempty iff the smallest $\mathbf{R}$-highset in S exists. In particular, we have $\bigcirc(S, \mathbf{R}) \neq \varnothing$ whenever S is a nonempty finite set.[17]

By "maximization of $\mathbf{R}$ in S," we mean identifying $\bigcirc(S, \mathbf{R})$. This is not only intuitive, but it is also consistent with the standard case (because $\bigcirc(S, \mathbf{R})$ reduces to $\max(S, \mathbf{R})$ when $\mathbf{R}$ is transitive). Furthermore, the following fact demonstrates that top-cycles indeed correspond to a well-defined optimization principle, thereby also clarifying that our definition is consistent with how top-cycles are traditionally defined in, say, social choice theory.

Proposition 4.1. *Let S be a nonempty subset of a set X, and $\mathbf{R}$ a complete binary relation on X. Then,*

$$\bigcirc(S, \mathbf{R}) = \max(S, \operatorname{tran}(\mathbf{R}|_S)).$$

Let us call a nonempty subset A of X an **$\mathbf{R}$-cycle** if for any x and y in A, there exist finitely many $a_1, ..., a_k$ in A such that $x \; \mathbf{R} \; a_1 \; \mathbf{R} \cdots \mathbf{R} \; a_k \; \mathbf{R} \; y$. (If A is finite and $\mathbf{R}$ is a complete binary relation on X, then A is an $\mathbf{R}$-cycle iff we can enumerate A as $\{x_1, ..., x_n\}$ such that $x_1 \; \mathbf{R} \; x_2 \; \mathbf{R} \cdots \mathbf{R} \; x_n \; \mathbf{R} \; x_1$.) As a matter of fact, we can use Proposition 4.1 to obtain yet another characterization of the top-cycle set provided that it is not empty. The next result identifies $\bigcirc(S, \mathbf{R})$ as the unique $\mathbf{R}$-highset in S that is also $\mathbf{R}$-cycle, hence justifying the term "top-cycle."[18]

Corollary 4.2. *Let S and T be nonempty subsets of X, and $\mathbf{R}$ a complete binary relation on X. Then, $T = \bigcirc(S, \mathbf{R})$ if, and only if, T is both an $\mathbf{R}$-highset in S and an $\mathbf{R}$-cycle.*

Proof. Suppose that $T = \bigcirc(S, \mathbf{R})$. Then, T is an $\mathbf{R}$-highset in S by definition of the top-cycle set. Moreover, by Proposition 4.1, if $x, y \in T$, then $x \operatorname{tran}(\mathbf{R}|_S)^{=} y$, and hence there exist $k \in \mathbb{N}$ and

[16] *Proof.* Suppose A and B are two $\mathbf{R}$-highsets in S with $A \subseteq B$ false. Then, pick any $a \in A \backslash B$, and notice that, for any $b \in B$, we have $b \; \mathbf{R}^{>} \; a$ because $a \in S \backslash B$ and B is an $\mathbf{R}$-highset in S. As A is itself an $\mathbf{R}$-highset in S, and a is in A, this implies $b \in A$ for each $b \in B$, that is, $B \subseteq A$.

[17] Mainly in the literature on voting theory, the notion of top cycles are studied extensively in the case where $\mathbf{R}$ is a tournament (that is, an asymmetric total binary relation on a finite set). See, for instance, Laslier (1997). When $\mathbf{R}$ is an arbitrary total binary relation, some authors refer to $\bigcirc(S, \mathbf{R})$ as the *weak top-cycle* of $\mathbf{R}$ in S (cf. Ehlers and Sprumont (2008)), and some as the $\mathbf{R}$-*admissible set* in S (cf. Kalai and Schmeidler (1977)).

[18] This result is not new; it was proved by Schwartz (1972) in the case where S is finite.

$a_1, \ldots, a_k$ in S such that $x\,\mathbf{R}\,a_1\,\mathbf{R}\cdots\mathbf{R}\,a_k\,\mathbf{R}\,y$. Therefore, T is an $\mathbf{R}$-cycle. Conversely, suppose that T is both an $\mathbf{R}$-highset in S and an $\mathbf{R}$-cycle. That T is an $\mathbf{R}$-highset in S implies $x\,\mathrm{tran}(\mathbf{R}|_S)^>\,y$ for any $x \in T$ and $y \in S\backslash T$, whereas that T is an $\mathbf{R}$-cycle implies $x\,\mathrm{tran}(\mathbf{R}|_S)^=\,y$ for all $x, y \in T$. Put together, we get $T = \max(S, \mathrm{tran}(\mathbf{R}|_S))$. In view of Proposition 4.1, we are done. ■

Existence of Top Cycles. Most works on top-cycles with respect to a complete binary relation take the ground set X of the binary relation as finite. This, in turn, makes the issue of existence of top-cycles a trivial matter. More generally, one can use basic topological hypotheses to guarantee the existence of top-cycles even when X is not finite. One such theorem will be proved in Section 4.3.

4.2 Rationalization by Preference Structures

We now turn to the primary inquiry of the present paper, namely, to the fundamental issue of defining how "choices" are made on the basis of a given preference structure. We wish to investigate this matter at a suitably general level (without restricting attention only to finite choice problems). To this end, let X be any nonempty set, and let $\mathfrak{X}$ be any collection of nonempty subsets of X such that (i) $\mathfrak{X}$ contains all singletons, and (ii) $\mathfrak{X}$ is closed under taking finite unions. (In particular, $\mathfrak{X}$ contains all nonempty finite subsets of X). For ease of reference, we will refer to any such ordered pair $(X, \mathfrak{X})$ as a **choice environment**. For example, $(X, 2^X\backslash\{\varnothing\})$ is a choice environment. More generally, where $\mathfrak{X}_{<\infty}$ denotes the collection of all nonempty finite subsets of X, $(X, \mathfrak{X}_{<\infty})$ is a choice environment; this is the environment used by the vast majority of works in the theory of individual choice. (In fact, this is the only choice environment when X is finite.) Still more generally, $(X, \mathbf{k}(X))$ is a choice environment, where X is a topological space and $\mathbf{k}(X)$ stands for the set of all nonempty compact subsets of X. We will obtain our main existence theorem in the context of this environment.

Given any choice environment $(X, \mathfrak{X})$, by a **choice correspondence on $\mathfrak{X}$**, we mean a set-valued map $C : \mathfrak{X} \rightrightarrows X$ such that $C(S) \subseteq S$ for every $S \in \mathfrak{X}$ and $C(S) \neq \varnothing$ for every finite $S \in \mathfrak{X}$. Such a choice correspondence C is said to be **single-valued** if $C(S)$ is a singleton for every finite $S \in \mathfrak{X}$.

Now take any weak preference structure $(\succsim, \mathbf{R})$ on X. We say that a choice correspondence C on $\mathfrak{X}$ is **rationalized by** $(\succsim, \mathbf{R})$ if

$$C(S) = \bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}), \tag{3}$$

or equivalently,

$$C(S) = \max(\mathbf{MAX}(S, \succsim), \mathrm{tran}(\mathbf{R}|_{\mathrm{MAX}(S,\succsim)})), \tag{4}$$

for every $S \in \mathfrak{X}$. Thus, we posit that an agent with a weak preference structure $(\succsim, \mathbf{R})$ settles on her choice(s) from a given feasible set S by employing a two-step procedure. First, she looks for those alternatives in S that are maximal with respect to her core preference relation $\succsim$. If there is only one such alternative in S, then she chooses that alternative. If there is a multiplicity of such alternatives (which may be due to indifferences and/or incomparabilities instigated by $\succsim$), then she restricts her attention to those alternatives, and evaluates them on the basis of her second (complete) binary relation $\mathbf{R}$. She finalizes her choice(s) by maximizing $\mathbf{R}$ on $\mathbf{MAX}(S, \succsim)$ in the sense of finding the top-cycle in $\mathbf{MAX}(S, \succsim)$ with respect to $\mathbf{R}$. This top-cycle is the set of all alternatives she deems "choosable" in S.

Replacing the top-cycle operator by an alternative maximization notion, such as the uncovered set (Lombardi (2008)) or the untrapped set (Duggan (2007)), would yield alternative theories of optimization. In what follows, we will derive a number of results that will hopefully witness that working with top-cycles leads to a particularly useful theory, but obviously, one cannot argue that our definition is the "right" one on *a priori* grounds.

Remark. After the important contribution of Manzini and Mariotti (2007) to choice theory, choice correspondences $C : \mathfrak{X} \rightrightarrows X$ of the form $C(S) = \max(\mathbf{MAX}(S, \mathbf{R}_1), \mathbf{R}_2)$, where $\mathbf{R}_1$ and $\mathbf{R}_2$ are binary relations on a finite set X, are often called *sequentially rationalized choice procedures.*[19] The spirit of the notion of choice correspondences rationalized by preference structures is certainly in concert with such choice procedures. However, despite the initial appearance of the formula (4), such a choice correspondence is, in general, not a sequentially rationalized choice procedure. This is because, for a feasible set $S \in \mathfrak{X}$, $\mathrm{tran}(\mathbf{R}|_{\mathrm{MAX}(S,\succsim)})$ is not the same relation as $\mathrm{tran}(\mathbf{R})$ in general. Thus, there is no "one" second binary relation used in a choice correspondence rationalized by a preference structure.[20]

4.3 Existence of Choice by Preference Structures

One of the main principles of optimization theory is the fact there is a maximum element in every compact space with respect to a continuous and complete preorder. In fact, it is known that one can even relax the transitivity requirement here, provided that we look for top-cycle elements instead of maxima. In this section, we show that these observations extend to the more general context of preference structures.

We say that a binary relation on a topological space X is **continuous** if it is a closed subset of $X \times X$ (relative to the product topology). In turn, a weak preference structure $(\succsim, \mathbf{R})$ is **continuous** if both $\succsim$ and $\mathbf{R}$ are continuous binary relations on X. Our main existence theorem says that the choice correspondence rationalized by such a preference structure is nonempty-valued on any compact subset of X.

Theorem 4.3. *For any topological space X, the choice correspondence on $\mathbf{k}(X)$ rationalized by a continuous weak preference structure $(\succsim, \mathbf{R})$ on X is nonempty-valued.*[21]

One of the major difficulties with working with nontransitive preferences in an economic setting is that even finite menus may not possess a maximal element with respect

[19] Strictly speaking, the model considered by Manzini and Mariotti (2007), the *sequential shortlisting method*, applies only to single-valued choice correspondences. The more general formulation we consider here was studied recently by García-Sanz and Alcantud (2015).

[20] Where X is finite, García-Sanz and Alcantud (2015) show that any sequentially rationalized choice correspondence C on $\mathfrak{X}_{<\infty}$ satisfies the so-called *Expansion Property*: $C(S) \cap C(T) \subseteq C(S \cup T)$ for every $S, T \in P(X)$. By contrast, a choice correspondence C on $\mathfrak{X}_{<\infty}$ that is rationalized by a preference structure need not satisfy this property. For instance, let $X := \{x_1, ..., x_5\}$, $\succsim := \triangle_X \sqcup \{(x_3, x_5), (x_4, x_2)\}$, and consider the complete binary relation $\mathbf{R}$ on X with $\mathbf{R}^> := \{(x_3, x_5), (x_4, x_2), (x_3, x_1), (x_4, x_1)\}$. Then, $(\succsim, \mathbf{R})$ is a preference structure on X. Now, put $S := \{x_1, x_2, x_3\}$ and $T := \{x_1, x_4, x_5\}$, and check that x_1 belongs to $C(S) \cap C(T)$, but not to $C(S \cup T)$.

[21] One can relax the continuity assumption to upper semicontinuity here; it is enough to assume in Theorem 4.3 that $x^{\uparrow,\succsim}$ and $x^{\uparrow,\mathbf{R}}$ are closed in X for every $x \in X$.

to such a preference. Theorem 4.3 shows that the present theory is free of this difficulty. Under the usual assumptions of compactness and continuity, there always exists a "choice" with respect to a preference structure, provided that we define choice by a top-cycle element (with respect to one's revealed preference) within the set of all maximal elements in a menu (with respect to that person's core preference).

In passing, we note that the earliest (topological) theorem on the existence of top cycles is due to Kalai and Schmeidler (1977). While that theorem applies to complete and continuous binary relations only under the hypothesis of antisymmetry, Duggan (2007) has shown that the antisymmetry requirement is in fact not needed. In turn, Theorem 4.3 generalizes Duggan's Theorem; the latter obtains from Theorem 4.3 simply by setting $\succsim$ as $\triangle_X$:

Corollary 4.4. [Duggan, 2007] *Let S be a nonempty compact subset of a topological space X, and $\mathbf{R}$ a continuous and complete binary relation on X. Then, $\bigcirc(S, \mathbf{R}) \neq \varnothing$.*

Remark. It may at first seem like Theorem 4.3 can be obtained by applying Duggan's Theorem to $\mathbf{MAX}(S, \succsim)$, where S is a compact subset of topological space X and $\succsim$ a continuous preorder on X. This is not correct. While continuity of $\succsim$ and compactness of S jointly entail that $\mathbf{MAX}(S, \succsim)$ is nonempty, easy examples would show that this set may fail to be compact. As such, Theorem 4.3 is a bit unexpected, and is proved by means of a direct argument without invoking Duggan's Theorem.

4.4 Equivalent Preference Structures

In the standard theory of rational choice, a choice correspondence can be rationalized by at most one complete preference relation (provided that the domain of the correspondence is rich enough). This is no longer true for choice correspondences that are rationalized by preference structures. That is, more than one preference structure may well rationalize a given choice correspondence; we may think of such structures as "equivalent" from the perspective of choice.

Definition. Given any choice environment $(X, \mathfrak{X})$, two weak preference structures $(\succsim, \mathbf{R})$ and $(\succsim', \mathbf{R}')$ on X are said to be **equivalent** if

$$\bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}) = \bigcirc(\mathbf{MAX}(S, \succsim'), \mathbf{R}')$$

for every $S \in \mathfrak{X}$; we denote this situation by writing $(\succsim, \mathbf{R}) \cong (\succsim', \mathbf{R}')$.[22]

In Section 3.3, we have stated that there is a "natural" way of pruning a weak preference structure to make it a proper preference structure that is indistinguishable from the former in terms of choice theory. The notion of *equivalence* helps formalize this point. To wit, take any weak preference structure $(\succsim, \mathbf{R})$ on X, and define the relation $\mathbf{R}_{\succsim}$ on X as in Section 3.3: $x \, \mathbf{R}_{\succsim} \, y$ iff either $x \succsim y$, or x and y are not $\succsim$-comparable and $x \, \mathbf{R} \, y$. Then, regardless of the choice environment, we have $(\succsim, \mathbf{R}) \cong (\succsim, \mathbf{R}_{\succsim})$. That is:

[22]It is plain that $\cong$ is an equivalence relation on the collection of all weak preference structures on X, but this relation depends on $\mathfrak{X}$. We do not use a notation that makes this dependence explicit only for brevity.

Proposition 4.5. *In the context of any choice environment* $(X, \mathfrak{X})$, *any weak preference structure* $(\succsim, \mathbf{R})$ *on* X *is equivalent to the* (*proper*) *preference structure* $(\succsim, \mathbf{R}_{\succsim})$.

Proof. Take any $S \in \mathfrak{X}$, and note that there is nothing to prove if there is no $\succsim$-maximal element in S. So, assume otherwise, and take any $\succsim$-maximal elements x and y of S. Then, either $x \sim y$ or $(x, y) \in \mathrm{Inc}(\succsim)$. In the former case, we have $x\,\mathbf{R}^{=}\,y$ and $x\,\mathbf{R}_{\succsim}^{=}\,y$ (because both $\mathbf{R}$ and $\mathbf{R}_{\succsim}$ are superrelations of $\succsim$), and in the latter case, $x\,\mathbf{R}\,y$ iff $x\,\mathbf{R}_{\succsim}\,y$ by definition of $\mathbf{R}_{\succsim}$. Thus, the restrictions of $\mathbf{R}$ and $\mathbf{R}_{\succsim}$ to $\mathbf{MAX}(S, \succsim)$ are the same. In view of the arbitrary choice of S, and Proposition 4.1, therefore, $(\succsim, \mathbf{R}) \cong (\succsim, \mathbf{R}_{\succsim})$. ∎

Let $(X, \mathfrak{X})$ be a choice environment, and recall that $\mathrm{w}\mathcal{PS}_X$ and $\mathcal{PS}_X$ stand for the set of all weak and proper preference structures on X, respectively, while the natural epimorphism $\pi : \mathrm{w}\mathcal{PS}_X \to \mathcal{PS}_X$ is defined by $\pi((\succsim, \mathbf{R})) := (\succsim, \mathbf{R}_{\succsim})$. Now, let $\mathrm{Choice}_{\mathfrak{X}}$ stand for the class of all choice correspondences on $\mathfrak{X}$, and define the map $c : \mathrm{w}\mathcal{PS}_X \to \mathrm{Choice}_{\mathfrak{X}}$ by setting $c((\succsim, \mathbf{R}))$ to be the choice correspondence on $\mathfrak{X}$ that is rationalized by $(\succsim, \mathbf{R})$. Then, Proposition 4.5 says that the following diagram, in which the restriction of c to $\mathcal{PS}_X$ is also denoted by c, commutes. In particular, π is a selection from the quotient map on $\mathrm{w}\mathcal{PS}_X$ relative to the equivalence relation $\cong$.

$$
\begin{tikzcd}
 & \mathrm{Choice}_{\mathfrak{X}} \\
\mathrm{w}\mathcal{PS}_X \arrow[ur, "\mathrm{c}"] \arrow[r, "\pi"'] & \mathcal{PS}_X \arrow[u, "\mathrm{c}"']
\end{tikzcd}
$$

Figure 4.1

In a nutshell, we conclude that a choice correspondence is rationalizable by a weak preference structure if, and only if, it is rationalizable by a proper preference structure.

4.5 Examples

The following examples are meant to illustrate the breadth of the notion of rationalization by preference structures. Unless otherwise is explicitly stated, $(X, \mathfrak{X})$ stands below for an arbitrarily fixed choice environment.

Example 4.1. (*The Rational Choice Model*) Let $\succsim$ be a complete preorder on X. Then, the choice correspondence C on $\mathfrak{X}$ rationalized by the transitive preference structure $(\succsim, \succsim)$ satisfies

$$C(S) = \max(S, \succsim) \quad \text{for every } S \in \mathfrak{X}.$$

Thus, the choice theory based on preference structures generalizes the standard choice theory that is based on complete preference relations.

Example 4.2. (*The Top-Cycle Choice Rule*) For a complete binary relation $\mathbf{R}$ on X, the choice correspondence C on $\mathfrak{X}$ rationalized by the preference structure $(\triangle_X, \mathbf{R})$ satisfies

$$C(S) = \bigcirc(S, \mathbf{R}) \quad \text{for every } S \in \mathfrak{X}.$$

Thus, the choice theory based on preference structures generalizes the theory of top-cycle choice rules that are commonly used in the theory of social choice and tournaments (cf. Kalai and Schmeidler (1977), Schwartz (1986), Laslier (1997), and Ehlers and Sprumont (2008).)

Example 4.3. (*The Undominated Choice Rule*) Let $\succsim$ be a preorder on X. Then, the choice correspondence C on $\mathfrak{X}$ rationalized by the preference structure $(\succsim, \succsim \sqcup \mathrm{Inc}(\succsim))$ satisfies

$$C(S) = \mathbf{MAX}(S, \succsim) \quad \text{for every } S \in \mathfrak{X}.$$

Thus, the choice theory based on preference structures generalizes the choice theory that is based on incomplete (but transitive) preference relations. (See, for instance, Eliaz and Ok (2006).)

Example 4.4. (*Pareto Refinement of Majority Voting*) Let $\mathcal{P}$ and $\mathcal{P}_{\mathrm{maj}}$ be defined as in Example 3.5. Then, the choice correspondence C on $\mathfrak{X}$ rationalized by the preference structure $(\bigcap \mathcal{P}, \mathcal{P}_{\mathrm{maj}})$ assigns to any feasible set $S \in \mathfrak{X}$ those Pareto optimal outcomes in S that maximizes the transitive closure of the majority voting rule on S. (Here, of course, Pareto optimality and majority voting rule are understood relative to the preference relations in $\mathcal{P}$.)

Example 4.5. (*Transitive Preference Structures*) We have noted in Section 4.2 that the choice correspondence rationalized by a preference structure is, in general, not a sequential choice procedure (in the sense of Manzini and Mariotti (2007)). However, the situation is different in the transitive case. To wit, let $(\succsim, \mathbf{R})$ be a transitive weak preference structure on a nonempty set X, and let C stand for the choice correspondence on $\mathfrak{X}$. Then,

$$C(S) = \max(\mathbf{MAX}(S, \succsim), \mathbf{R}) \tag{5}$$

for every $S \in \mathfrak{X}$. This sits square with the interpretation that $\succsim$ is the "sure" preferences of a person, and $\mathbf{R}$ corresponds to the rational (complete) preferences of a consultant. When dealing with a choice problem S, this person first identifies the undominated alternatives in S with respect to her inherent (core) preference relation $\succsim$. If there are more than one such alternative in S, then she is conflicted as to which of these to choose. In that case, she presents her reduced choice problem $\mathbf{MAX}(S, \succsim)$ to her consultant who identifies the best alternatives within $\mathbf{MAX}(S, \succsim)$ according to her own preferences, and our principal agent chooses (one of those) alternatives.

At the end of Section 3.3, we have noted that "choice-by-consultation" would be better modeled by means of the image of $(\succsim, \mathbf{R})$ under the natural epimorphism, that is, by $(\succsim, \mathbf{R}_{\succsim})$. The latter model is a proper preference structure, but it need not be transitive (Section 3.3). Nonetheless, we still have

$$C(S) = \max(\mathbf{MAX}(S, \succsim), \mathbf{R}_{\succsim}) \tag{6}$$

for every $S \in \mathfrak{X}$ with $C(S) \neq \varnothing$. (Indeed, for any such S, we have seen in the proof of Proposition 4.5 that $\mathbf{R}$ and $\mathbf{R}_{\succsim}$ agree on $\mathbf{MAX}(S, \succsim)$. As $\mathbf{R}$ is transitive here, therefore, $\mathbf{R}_{\succsim}$ is transitive on $\mathbf{MAX}(S, \succsim)$, so equation (6) follows from Proposition 4.1.)

Remark. In the context of Example 4.5, we have $C = \max(\cdot, \mathbf{R})$, provided that $(\succsim, \mathbf{R})$ is a transitive *proper* preference structure. For transitive *weak* preference structures, neither $C \subseteq \max(\cdot, \mathbf{R})$ nor $C \supseteq \max(\cdot, \mathbf{R})$ is, in general, true. The main problem is with the latter containment, however. To wit, when X is finite, we have $C \subseteq \max(\cdot, \mathbf{R})$, but easy examples would show that the converse containment need not hold.[23]

Example 4.6. (*The Constant Threshold Choice Model*) Let $u : X \to \mathbb{R}$ be any function and take any real number $\varepsilon \geq 0$. Define the binary relation $\mathbf{R}$ on X as $x \mathbf{R} y$ iff $u(x) \geq u(y) - \varepsilon$. Consider first the preorder $\succsim'$ on X defined by $x \succsim' y$ iff either $x = y$ or $u(x) > u(y)$. Then, $(\succsim', \mathbf{R})$ is a weak preference structure on X, and the choice correspondence C on $\mathfrak{X}$ rationalized by this preference structure is the rational choice model: $C(S) = \arg\max\{u(x) : x \in S\}$ for every $S \in \mathfrak{X}$. Next, consider the preorder $\succsim$ on X defined by $x \succsim y$ iff either $x = y$ or $u(x) > u(y) + \varepsilon$. Then, $(\succsim, \mathbf{R})$ is a preference structure on X (Example 3.7), and it is easy to prove that the choice correspondence C on $\mathfrak{X}$ rationalized by $(\succsim, \mathbf{R})$ satisfies

$$C(S) = \{x \in S : \sup u(S) - u(x) \leq \varepsilon\}$$

for every $S \in \mathfrak{X}$. Following Luce (1956), such a correspondence is referred to as a *constant threshold choice model.* Conclusion: Every constant threshold choice model is rationalized by a preference structure.

4.6 On the Predictive Power of the Model

The examples above illustrate that quite a number of choice models are captured by choice correspondences that are rationalized by preference structures. However, the predictive power of the present choice model is far from nil. Certainly not all choice correspondences arise from the maximization of a preference structure. Indeed, we have already seen in Section 2.2 that this model does not contain many sequential choice procedures. The following example provides a more direct, and simpler, demonstration.

Example 4.7. Put $X := \{x, y, z\}$, and take any choice correspondence C on $\mathfrak{X}_{<\infty}$ such that

$$x \in C\{x, y\} \quad \text{and} \quad \{y\} = C\{x, y, z\}.$$

To derive a contradiction, suppose C is a choice correspondence rationalized by a preference structure $(\succsim, \mathbf{R})$ on X. Since $y \in C\{x, y, z\}$, y is $\succsim$-maximal in X, and hence in $\{x, y\}$, so $x \in C\{x, y\}$ implies that $x \mathbf{R} y$ (Proposition 4.1). Since x does not belong to $C\{x, y, z\}$ but y does, therefore, x is not $\succsim$-maximal in X. As $y \succ x$ cannot hold (because $x \in C\{x, y\}$), we thus have $z \succ x$. Then, $y \succ z$ cannot hold, because otherwise

[23]We use the finiteness hypothesis here only for simplicity. In general, if $(\succsim, \mathbf{R})$ and $\mathfrak{X}$ are such that for every $S \in \mathfrak{X}$ and $y \in S \backslash \mathbf{MAX}(S, \succsim)$, there is an $x \in \mathbf{MAX}(S, \succsim)$ with $x \succ y$, then $C \subseteq \max(\cdot, \mathbf{R})$ holds. This is so, for instance, if X is topological space, $\mathfrak{X} = \mathbf{k}(X)$, and $\succsim$ is continuous (Lemma A.2).

$y \succ x$ and hence $\{y\} = C\{x, y\}$, a contradiction. It follows that z is $\succsim$-maximal in X, and hence $y \ \mathbf{R}^{>} \ z$ (because $\{y\} = C\{x, y, z\}$). Thus, $z \succ x \ \mathbf{R} \ y$ and $y \ \mathbf{R}^{>} \ z$, which contradicts $\succsim$-transitivity of $\mathbf{R}$.

Weak Axiom of Revealed Preference. Let X be any nonempty set, and recall that $\mathfrak{X}_{<\infty}$ stand for the collection of all nonempty finite subsets of X. The most well-known rationality criterion for a choice correspondence C on $\mathfrak{X}_{<\infty}$ is the Weak Axiom of Revealed Preference (WARP). This property can be decomposed into two distinct parts, known as, Property α (or the *Chernoff Axiom*) and Property β. We say that C on $\mathfrak{X}_{<\infty}$ satisfies the **Property** α if for every $S, T \in \mathfrak{X}_{<\infty}$ with $S \subseteq T$,

$$C(T) \cap S \subseteq C(S),$$

and that it satisfies **Property** β if for every $S, T \in \mathfrak{X}_{<\infty}$ with $S \subseteq T$,

$$x, y \in C(S) \quad \text{and} \quad x \in C(T) \quad \text{imply} \quad y \in C(T),$$

and finally, that it satisfies WARP if it obeys both of these properties. (Obviously, WARP and Property α are identical axioms for *single-valued* choice correspondences.) A fundamental result of choice theory says that C satisfies WARP iff it is rationalizable by a complete preference relation (in the sense that there exists a total preorder $\succsim$ on X with $C(S) = \max(S, \succsim)$ for every $S \in \mathfrak{X}_{<\infty}$).

Unsurprisingly, a choice correspondence rationalized by a preference structure need not satisfy either Property α or Property β. After all, choice correspondences of the form considered in Examples 4.2 and 4.3 are well-known to fail Properties α and β, respectively.

Remark. It is an easy exercise to verify that a choice correspondence on $\mathfrak{X}_{<\infty}$ that is rationalized by transitive weak preference structure satisfies Property α. However, such a choice correspondence may still fail Property β. (Indeed, for any preorder $\succsim$ on X, $\mathbf{MAX}(\cdot, \succsim)$ is a choice correspondence rationalized by the transitive weak preference structure $(\succsim, X \times X)$ on X.)

Single-Valued Choice Correspondences. While rationalization by a preference structure ensures, in general, neither Property α nor Property β, it turns out that it reduces to the standard notion of rationalization in the context of *single-valued* choice correspondences defined on finite choice problems. This is the content of our next proposition.

Proposition 4.6. *Let X be any nonempty set, and C a single-valued choice correspondence on $\mathfrak{X}_{<\infty}$. Then, C is rationalized by a weak preference structure if, and only if, it satisfies WARP.*

Proof. The "if" part of the claim is straightforward. Conversely, suppose C is rationalized by a weak preference structure $(\succsim, \mathbf{R})$ on X. Take any $S, T \in \mathfrak{X}_{<\infty}$ with $S \subseteq T$, and let $x \in S$ be such that $\{x\} = C(T)$. Then, $x \in \mathbf{MAX}(T, \succsim)$ and $x \ \mathbf{R} \ \mathbf{MAX}(T, \succsim)$. To derive a contradiction, suppose x does not belong to $C(S)$. Since $S \subseteq T$, we have $x \in \mathbf{MAX}(S, \succsim)$ so there must then exist a $y \in \mathbf{MAX}(S, \succsim)$ with $y \ \mathbf{R}^{>} \ x$. Since y does not belong to $C(T)$, therefore, y cannot be $\succsim$-maximal in T. Since T is finite, this means that $z \succ y$ for some $z \in \mathbf{MAX}(T, \succsim)$. Then, by $\succsim$-transitivity of $\mathbf{R}$, we find $z \ \mathbf{R} \ x$. By the choice of x, this is possible only if $z = x$. But this implies $x \succ y$, contradicting $\succsim$-maximality of y in S. Conclusion: C satisfies Property α (and hence WARP). ■

Proposition 4.6 is another demonstration of the predictive power of rationalization by preference structures. Indeed, it shows that the choice theory that is based on preference structures reduces to the standard theory of rational choice in the context of single-valued choice correspondences on finite choice problems. (Thus, for instance, the rational shortlisting models of Manzini and Mariotti (2007), Au and Kawai (2011), and Cherepanov et al. (2013), as well as the attention/competition filter models of Masatlioglu, Nakajima and Ozbay (2012) and Lleras, et al. (2017), and the attraction effect models of Lombardi (2009) and Ok, Ortoleva and Riella (2015), are not captured by this theory.) This is not really surprising. After all, the main goal of the model of preference structures is to capture behavioral traits such as indecisiveness and cyclic choices, and as such, the choice theory that this model induces is primed to make many-valued choice predictions.

The Condorcet Criterion. Let $(X, \mathfrak{X})$ be a choice environment. A choice correspondence C on $\mathfrak{X}$ is said to satisfy the **Condorcet Criterion** if for every $S \in \mathfrak{X}$ and $x \in S$,

$$x \in C\{x, y\} \text{ for every } y \in S \quad \text{imply} \quad x \in C(S).$$

The choice behavior that is rationalizable by a preference structure is sure to be consistent with this property.

Proposition 4.7. *Let C be a choice correspondence on $\mathfrak{X}$ rationalized by a weak preference structure $(\succsim, \mathbf{R})$ on X. Then, C satisfies the Condorcet Criterion.*

Proof. Take any $S \in \mathfrak{X}$, and let x be an element of S such that $x \in C\{x, y\}$ for every $y \in S$. Then, $x \in \mathbf{MAX}(S, \succsim)$ so, given that $C(S)$ is the top-cycle in $\mathbf{MAX}(S, \succsim)$ with respect to $\mathbf{R}$, if x did not belong to $C(S)$, we would have $y \; \mathbf{R}^{>} \; x$ for some $y \in \mathbf{MAX}(S, \succsim)$. But then, for this y, we would have $\{y\} = C\{x, y\}$, a contradiction. Conclusion: C satisfies the Condorcet Criterion. ■

Proposition 4.7 can also be used to distinguish those choice correspondences rationalized by preference structures from some of the boundedly rational choice correspondences introduced in the recent literature on choice theory. For instance, we see readily that the reference-dependent choice model of Ok, Ortoleva and Riella (2015) is distinct from the present choice model as the former does not satisfy the Condorcet Criterion. Similarly, the model of choice with limited consideration, introduced recently by Lleras et al. (2017), is distinct from the present choice model. A choice correspondence C on $\mathfrak{X}$ is said to be a *choice correspondence with limited consideration* if $C = \max(\Gamma(\cdot), \trianglerighteq)$ where $\trianglerighteq$ is a complete preorder on X, and Γ is a choice correspondence on $\mathfrak{X}$ that satisfies Property α. (Lleras et al. (2017) refer to Γ as a *competition filter* on $\mathfrak{X}$.) This model appears at first to be more general than the present choice model. But, in general, a choice correspondence with limited consideration does not satisfy the Condorcet Criterion, so it need not be rationalizable by a weak preference structure.[24] Similarly, justifiable choice

[24]Note that $\mathbf{MAX}(\cdot, \succsim)$ is a competition filter on $\mathfrak{X}$. Thus, $\max(\mathbf{MAX}(\cdot, \succsim), \mathrm{tran}(\mathbf{R}))$ is actually a choice correspondence on $\mathfrak{X}$ with limited consideration. However, this correspondence is in general distinct from $\bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R})$, and indeed, easy examples would show that it need not be rationalizable by a weak preference structure.

correspondences are, in general, not rationalizable by a weak preference structure.[25]

Finally, note that Example 4.6 shows that certain types of *satisficing* rules à la Herbert Simon are captured by the present choice theory. But not all threshold choice models, let alone all satisficing rules, are rationalizable by preference structures. For example, consider the choice correspondence C on $\mathfrak{X}$ defined by

$$C(S) = \{x \in S : \sup u(S) - u(x) \leq \varepsilon(S)\},$$

where u is any real map on X and $\varepsilon : \mathfrak{X} \to \mathbb{R}_+$ is a function such that $\varepsilon(A) \leq \varepsilon(B)$ for every $A, B \in \mathfrak{X}$ with $A \subseteq B$. (Where X is finite, Frick (2016) has recently provided an axiomatization of such choice correspondences, and called them *monotone threshold choice models*.) It is not difficult to show that such a model need not be rationalizable by a preference structure.

Characterization of Rationalization by Preference Structures. It is of great interest to provide a behavioral axiomatic characterization of all choice correspondences that are rationalizable by a preference structure. This would formally describe the predictive content of such correspondences fully, and provide a complete comparison of them with rational choice correspondences. Given the sizable length of the present paper, however, we do not take up this exercise here. Instead, this problem is tackled in a companion paper by Evren, Nishimura and Ok (2019), where a complete characterization is provided in the context of the choice environment $(X, \mathfrak{X}_{<\infty})$.

Other Behavioral Consistency Properties. Suppose C is a rationalizable choice correspondence on $\mathfrak{X}$ in the classical sense, that is, $C = \max(\cdot, \succsim)$ for some complete preorder $\succsim$ on X. Let z be a choice from a feasible set S by a decision maker whose choice behavior is modeled by this choice correspondence. If this agent is instead offered the feasible set $S \cup \{x\}$ where x is a new alternative at least as good as z, she would surely deem x choosable from this set: $x \in C(S \cup \{x\})$. It is in this sense that C is *monotonic* with respect to $\succsim$.

Now let C be the choice correspondence rationalized by some preference structure $(\succsim, \mathbf{R})$ on X. We would like to carry out the same query in this case as well, but now monotonicity may be checked either with respect to the core preference relation $\succsim$ of the agent, or her revealed preferences $\mathbf{R}$. Let us first look into the first situation. Suppose $z \in C(S)$ for some $S \in \mathfrak{X}$. Then, if the agent has no doubt in her mind that x is a better alternative than z, that is, $x \succsim z$, one would expect she view x as choosable from $S \cup \{x\}$. The following proposition shows that C possesses this property indeed.

Proposition 4.8. *Let C be a choice correspondence on $\mathfrak{X}$ rationalized by a weak preference structure $(\succsim, \mathbf{R})$ on X. Then, for any $S \in \mathfrak{X}$,*

$$x \succsim z \in C(S) \quad \textit{implies} \quad x \in C(S \cup \{x\}).$$

[25] A choice correspondence C on $\mathfrak{X}$ is said to be *justifiable* if there exists a nonempty collection $\mathcal{P}$ of complete preorders on X such that $C(S) = \bigcup\{\max(S, \succsim) : \succsim \in \mathcal{P}\}$ for every $S \in \mathfrak{X}$. (See, for instance, Heller (2012) and Costa, Ramos and Riella (2019).) To prove our claim, let $X := \{x, y, z\}$, and consider the choice correspondence C on $P(X)$ with $C(X) = \{x, z\}$ and $C(S) = S$ for each nonempty proper subset S of X. Then, C fails the Condorcet Criterion, but it is justifiable; choose $\mathcal{P}$ as $\{\succsim_1, \succsim_2\}$ with $x \succ_1 y \succ_1 z$ and $z \succ_2 y \succ_2 x$.

Let us now ask the same question with respect to the revealed preference relation $\mathbf{R}$. That is, suppose $z \in C(S)$ for some $S \in \mathfrak{X}$, and that we have observed the agent choose x over z (at least once). Would this agent necessarily deem x choosable from $S \cup \{x\}$? This is less clear than the previous situation. The decision maker may have chosen x over z with serious difficulty, perhaps referring to the preferences of another individual. Thus, it is possible that some alternatives in S may dominate x, but not z, with respect to the core preferences of the agent, and this may cause x be not chosen from $S \cup \{x\}$ even though z is deemed choosable from S. This may indeed be the case. However, if z is the *only* choice from S, the following proposition shows that C acts still monotonically with respect to $\mathbf{R}$.

Proposition 4.9. *Let C be a choice correspondence on $\mathfrak{X}$ rationalized by a preference structure $(\succsim, \mathbf{R})$ on X. Then, for any finite $S \in \mathfrak{X}$,*

$$x \mathbf{R} z \text{ and } \{z\} = C(S) \quad \text{imply} \quad x \in C(S \cup \{x\}).$$

Remark. The requirement "$\{z\} = C(S)$" cannot be relaxed to "$z \in C(S)$" in Proposition 4.9. To see this, let $X := \{x, y, z\}$, and define $\succsim := \triangle_X \sqcup \{(y, x)\}$ and $\mathbf{R} := X^2 \backslash \{(x, y)\}$. Then, $(\succsim, \mathbf{R})$ is a preference structure on X. But for the choice correspondence C on $\mathfrak{X}_{<\infty}$ rationalized by $(\succsim, \mathbf{R})$, we have $C(X) = \{y, z\} = C\{y, z\}$ while $x \mathbf{R} z$.

It is also worth noting that Proposition 4.9 is not valid for weak preference structures. For instance, if $X := \{x, y, z\}$, $\succsim := \triangle_X \sqcup \{(z, x), (z, y)\}$ and $\mathbf{R} := X \times X$. Then, $(\succsim, \mathbf{R})$ is a weak preference structure on X, and $x \mathbf{R} z$ and $\{z\} = C\{y, z\}$, but x does not belong to $C\{x, y, z\}$.

5 Elicitation of Preference Structures

Rational choice theory is built on the hypothesis that the choice correspondence of a rational individual arises from the maximization of a complete preorder. Moreover, such a choice correspondence is rationalized by a *unique* complete preorder. (That is, if $(X, \mathfrak{X})$ is a choice environment and $\max(\cdot, \succsim) = \max(\cdot, \succsim')$ on $\mathfrak{X}$ for some complete preorders $\succsim$ and $\succsim'$ on X, then $\succsim = \succsim'$.) Thus, every complete preorder (interpreted as a preference relation) gives rise to a unique rationalizable choice model, and every rationalizable choice model induces a unique preference relation (that arises from pairwise choice problems). While trivial, this duality is an essential aspect of rational choice theory.

In this section, we investigate if, and how, such a duality exists for the choice model induced by preference structures. That is, we examine the relation between two preference structures that happen to rationalize the same choice correspondence. Put a bit more precisely, we would like to understand exactly how two *equivalent* preference structures $(\succsim, \mathbf{R})$ and $(\succsim', \mathbf{R}')$ relate to each other. The exact analogue of the situation in rational choice theory would be to have $\succsim = \succsim'$ and $\mathbf{R} = \mathbf{R}'$ in this instance. The second of these equations is indeed correct (provided that we work with *proper* preference structures), but mainly because different preorders with the same asymmetric part would declare the same elements as maximal in all feasible sets, the first equation is, in general, false. (For instance, $(\triangle_X, X \times X)$ and $(X \times X, X \times X)$ rationalize the same choice correspondence.) However, we will show below that one can always identify "the" largest preference structure – this is the one whose core preference exhibits the least amount of incompleteness – that rationalizes a choice correspondence which is known to be rationalizable by *some* preference structure. Thus, the present choice model too exhibits a useful, and this time entirely nontrivial, duality. Every preference structure gives rise to a unique rationalizable choice model (in the sense of (3) and in the context of a suitably general choice environment), and conversely, every choice model that is rationalizable by a preference structure induces a unique largest preference structure.

5.1 Elicitation in General Choice Environments

Revealed Preferences. Let $(X, \mathfrak{X})$ be any choice environment. Let C be a choice correspondence on $\mathfrak{X}$. There are a number of important preference relations on X that we may define by using C. Perhaps the most obvious is the binary relation $\mathbf{R}_C$ on X defined by

$$x \, \mathbf{R}_C \, y \quad \text{iff} \quad x \in C\{x, y\}.$$

In words, $x \, \mathbf{R}_C \, y$ means that the agent (with choice correspondence C) would choose x over y when comparing these two alternatives alone. Naturally, we refer to $\mathbf{R}_C$ as the **revealed preference relation** induced by C. This relation is complete (because C is nonempty-valued over finite sets).

The following elementary observation highlights the importance of $\mathbf{R}_C$.

Lemma 5.1. *Let $(X, \mathfrak{X})$ be any choice environment, and C a choice correspondence on $\mathfrak{X}$. If C is rationalized by a preference structure $(\succsim, \mathbf{R})$ on X, then $\mathbf{R} = \mathbf{R}_C$.*

Proof. For any x and y in X, setting $S = \{x, y\}$ in (3) yields $\{x\} = C\{x, y\}$ iff $x \, \mathbf{R}^{>} \, y$, and $\{x, y\} = C\{x, y\}$ iff $x \, \mathbf{R}^{=} \, y$. Thus: $\mathbf{R} = \mathbf{R}_C$. ■

In other words, the revealed preferences of an individual whose choices are rationalized by a preference structure are uniquely identified from her binary choice decisions. This shows that the interpretation of preference structures we outlined in Section 3.1 is duly consistent with the choice theory we introduced in Section 4.

We should emphasize that Lemma 5.1 is not valid for *weak* preference structures. Indeed, if $(\succsim, \mathbf{R})$ is weak preference structure on X, then $(\succsim, \mathbf{R}) \cong (\succsim, \mathbf{R}_{\succsim})$ by Proposition 4.1, but, in general, $\mathbf{R}$ and $\mathbf{R}_{\succsim}$ are distinct relations. Thus, at least from the perspective of preference identification, proper preference structures appear to be superior to weak preference structures.

Revealed Core Preferences. Unlike its revealed part, the core preference part of a preference structure is not observable by an outside observer, so it is particularly important to understand which sorts of preorders on X rationalize a given choice correspondence C on $\mathfrak{X}$ when coupled with $\mathbf{R}_C$. We denote the set of all such preorders by $\mathbb{P}(C)$, that is,

$$\mathbb{P}(C) := \{\succsim \, : \, (\succsim, \mathbf{R}_C) \text{ is a preference structure on } X \text{ that rationalizes } C\}.$$

Thus: $\mathbb{P}(C) \neq \varnothing$ *iff* C *is rationalizable by a preference structure.* Moreover, for any $\succsim$ and $\succsim'$ in $\mathbb{P}(C)$, we have $(\succsim, \mathbf{R}_C) \cong (\succsim', \mathbf{R}_C)$, that is, choice-theoretically, there is no difference between $\succsim$ and $\succsim'$.

There is a natural way of partially ordering all preorders on X on the basis of their incompleteness. For any two such preorders $\succsim$ and $\succsim'$, we say that $\succsim$ is **more complete than** $\succsim'$ if the former preorder extends the latter. We denote this partial order (as well as any restriction of it to a given set of preorders on X) by $\sqsupseteq$. In particular, for any $\succsim$ and $\succsim'$ in $\mathbb{P}(C)$, we have

$$\succsim \;\sqsupseteq\; \succsim' \quad \text{iff} \quad \succsim \text{ extends } \succsim' .$$

Clearly, endowing $\mathbb{P}(C)$ with $\sqsupseteq$ makes it a poset. In fact, even at this level of generality, this poset possesses a nontrivial structure:

Theorem 5.2. *Let $(X, \mathfrak{X})$ be any choice environment, and C a choice correspondence on $\mathfrak{X}$. If C is rationalized by at least one preference structure on X, then $\mathbb{P}(C)$ is a complete $\vee$-semilattice.*[26]

This result, whose proof is somewhat involved, may at first appear as a technical observation, but, in fact, it provides a clear insight about those core preferences that rationalize a given choice correspondence C when coupled with the revealed preference relation induced by C. Apparently, any collection of such core preference relations can be combined to get another, more decisive, core preference that still rationalizes C when adjoined to $\mathbf{R}_C$. In particular, there is a *most decisive* core preference on X. (We will compute this preference shortly.)

Remark. $\mathbb{P}(C)$ need not be a $\wedge$-semilattice under the hypotheses of Theorem 5.2. To see this, pick any two objects outside $\mathbb{N}^2$, say, a and b, and put $X := \mathbb{N}^2 \sqcup \{a, b\}$. Next, take the partial order $\succeq_{\mathrm{I}}$ on X with $x \succ_{\mathrm{I}} y$ iff either $x, y \in \mathbb{N}^2$, $x_1 > y_1$ and $x_2 = y_2$, or $x \in \mathbb{N}^2$ and $y = a$. Similarly, let $\succeq_{\mathrm{II}}$ stand for the partial order on X with $x \succ_{\mathrm{II}} y$ iff either $x, y \in \mathbb{N}^2$, $x_1 = y_1$ and $x_2 > y_2$, or $x \in \mathbb{N}^2$ and $y = a$. Finally, define the binary relation $\mathbf{R}$ on X as $x \mathbf{R} y$ iff either $x, y \in \mathbb{N}^2$ and $x_1 + x_2 \geq y_1 + y_2$, or $y \in \{a, b\}$. Then, $(\succeq_{\mathrm{I}}, \mathbf{R})$ and $(\succeq_{\mathrm{II}}, \mathbf{R})$ are transitive preference structures on X. Now, put $\mathfrak{X} := \mathfrak{X}_{<\infty} \cup \{X\}$, and note that $(X, \mathfrak{X})$ is a choice environment. Beides, both $(\succeq_{\mathrm{I}}, \mathbf{R})$ and $(\succeq_{\mathrm{II}}, \mathbf{R})$ rationalize the choice correspondence C on $\mathfrak{X}$, where $C(S) = \max(S, \mathbf{R})$ for any $S \in \mathfrak{X}_{<\infty}$ and $C(X) = \{b\}$. Thus: $\{\succeq_{\mathrm{I}}, \succeq_{\mathrm{II}}\} \subseteq \mathbb{P}(C)$. But, if $\succsim$ is a preorder on X with $\succeq_{\mathrm{I}} \supseteq \succsim$ and $\succeq_{\mathrm{II}} \supseteq \succsim$, then no two elements of $\mathbb{N}^2$ are $\succsim$-comparable, so $\mathbf{MAX}(X, \succsim) \supseteq \mathbb{N}^2$, and hence, $\bigcirc(\mathbf{MAX}(X, \succsim), \mathbf{R}) = \varnothing$, which means that $(\succsim, \mathbf{R})$ does not rationalize C. Thus, there is no infimum of $\succeq_{\mathrm{I}}$ and $\succeq_{\mathrm{II}}$ in $\mathbb{P}(C)$ relative to $\sqsupseteq$.

The Largest Revealed Core Preference. The revealed preference relation induced by C arises only through pairwise comparisons of alternatives, and it does not tell us whether or not $x \mathbf{R}_C y$ entails the choosability of x over y across all feasible sets. This task would be handled by those preorders $\succsim$ that are compatible with $\mathbf{R}_C$ (in the sense that C is rationalized by $(\succsim, \mathbf{R}_C)$). Indeed, according to our interpretation of $(\succsim, \mathbf{R}_C)$, $x \succ y$ means that the agent prefers x over y "obviously," so it stands to reason that she would never choose y in any situation in which x is feasible; the presence of x in

[26] A poset $(A, \trianglerighteq)$ is said to be a *complete* $\vee$*-semilattice* if for every nonempty subset B of A, the $\trianglerighteq$-minimum of the set $\{a \in A : a \trianglerighteq B\}$ exists; *complete* $\wedge$*-semilattices* are defined dually. If $(A, \trianglerighteq)$ is both a complete $\vee$-semilattice and a complete $\wedge$-semilattice, we say that it is a *complete lattice*.

any menu rules out y being a potential choice. This prompts looking at the asymmetric binary relation $\succ_C$ on X with

$$x \succ_C y \quad \text{iff} \quad y \notin C(S) \text{ for every } S \in \mathfrak{X} \text{ with } x \in S.$$

We refer to $\succ_C$ as the **revealed core dominance** induced by C.

It is also possible that x and y are "obviously" equally appealing for the decision maker. From the vantage point of choice theory, this means that replacing x with y in any choice problem does not at all alter the choice behavior of the agent (apart from the replacement of x with y). That is, if x is deemed choosable (or unchoosable) in a menu, replacing x with y in that menu would yield a menu in which y is choosable (or, respectively, unchoosable). In addition, the choosability status of any other alternative in the two menus remains the same. We are thus led to define the binary relation $\sim_C$ on X by $x \sim_C y$ iff

$$x \in C(S \cup \{x\}) \text{ iff } y \in C(S \cup \{y\})$$

and

$$z \in C(S \cup \{x\}) \text{ iff } z \in C(S \cup \{y\})$$

for every $S \in \mathfrak{X}$ and every $z \in S$. Put succinctly, $x \sim_C y$ means that x and y are perfect substitutes in that replacing one for the other does not change the choice behavior of the agent in any choice situation. This relation, which we borrow from Riberio and Riella (2017), is called the **revealed core indifference** induced by C. It is a symmetric relation disjoint from $\succ_C$.

Finally, we define $\succsim_C$ as the union of the relations $\succ_C$ and $\sim_C$, and refer to it as the **revealed core preference relation** induced by C. In general, $\succsim_C$ is a subrelation of $\mathbf{R}_C$ distinct from $\mathbf{R}_C$. And, in fact, not only is $(\succsim_C, \mathbf{R}_C)$ is a preference structure on X that rationalizes C, but it is the largest such structure. Put differently, $\succsim_C$ turns out to be the top element of the complete $\vee$-semilattice $\mathbb{P}(C)$. This is the second main result of this section:

Theorem 5.3. *Let $(X, \mathfrak{X})$ be any choice environment, and C a nonempty-valued choice correspondence on $\mathfrak{X}$. If C is rationalized by at least one preference structure on X, then*

$$\bigvee \mathbb{P}(C) = \succsim_C.$$

Once again, this is not meant to be a technical result. Rather, it characterizes the most decisive core preference relation compatible with a choice correspondence (that is rationalizable by a preference structure). While the core part of a preference structure is, in general, not observable and non-unique, we can still elicit this part, in its most decisive form, from one's (observable) choice behavior. In the next section, we will show that, in most cases of interest, this computation is fairly straightforward. For now, we provide a concrete example.

Example 5.1. (*Revealed Preferences with Imperfect Discrimination*) Take any integer $n \geq 2$, and let $u : \mathbb{R}^n \to \mathbb{R}$ be a continuous surjection. Pick any $\varepsilon > 0$, and consider the preference structure $(\succsim, \mathbf{R})$ on $\mathbb{R}^n$ where $x \mathbf{R} y$ iff $u(x) \geq u(y) - \varepsilon$, and $x \succsim y$ iff either

$x = y$ or $u(x) > u(y) + \varepsilon$. Let C be the choice correspondence on $\mathbf{k}(\mathbb{R}^n)$ rationalized by $(\succsim, \mathbf{R})$. Using the characterization of C given in Example 4.5, one can show that

$$x \succsim_C y \quad \text{iff} \quad \text{either } u(x) = u(y) \text{ or } u(x) > u(y) + \varepsilon$$

for any $x, y \in \mathbb{R}^n$. (Proof is left as an exercise.) So, in this case, we have $\succ \, = \, \succ_C$, but (since u cannot be injective), we have $\succsim \, \neq \, \succsim_C$.

Remark. For expositional purposes, we have not stated Theorem 5.3 above in its strongest form. It turns out that the nonempty-valuedness hypothesis can be omitted in the statement of this theorem, but at the cost of lengthening the proof significantly. In view of Theorem 4.3, however, this hypothesis is largely inconsequential for applications.

5.2 Elicitation in Compact Choice Environments

While it applies in *any* choice environment, Theorem 5.2 has a shortcoming. The ordering of $\mathbb{P}(C)$ used in this result exhibits a somewhat unnatural asymmetry: While any two rationalizing core preferences can be combined to get a more decisive such preference, we cannot, in general, find a less decisive such preference. Our final theorem shows that if the alternative space has a topology, and we restrict our attention to compact choice problems, then this difficulty disappears, under a standard continuity condition.

Theorem 5.4. *Let X be a topological space, and let C be the choice correspondence on $\mathbf{k}(X)$ rationalized by a continuous preference structure $(\succsim, \mathbf{R})$ on X. Then, $\mathbb{P}(C)$ is a complete lattice, and $\bigvee \mathbb{P}(C) = \succsim_C$.*[27]

Taking stock, the notions of a complete preference relation and a rationalizable choice correspondence (in the classical sense) are duly, and trivially, consistent with each other. Theorem 5.4 establishes a similar duality between the notions of a preference structure and a choice correspondence rationalizable by a preference structure. This time things are nontrivial, because for every such choice correspondence C, there is a whole collection of core preferences (but a single revealed preference) that are compatible with C. But, under standard (topological) conditions, that collection is a complete lattice (relative to the "more complete than" ordering), so there is an inherent discipline to the model. In particular, so long as we pick the revealed core preference and revealed preference relations induced by C as "representative," then the models $(\succsim_C, \mathbf{R}_C)$ and C stand dual to each other, provided that C is rationalizable by a preference structure. (That is, $(\succsim_C, \mathbf{R}_C)$ is defined through C, and the choice correspondence that $(\succsim_C, \mathbf{R}_C)$ rationalizes is precisely C.)

Remark. It may be of interest to see what $\bigwedge \mathbb{P}(C)$ looks like; this is the *least decisive* core preference that rationalizes C when combined with $\mathbf{R}_C$. This relation is found as $\mathrm{tran}(\rhd) \cup \triangle_X$, where $\rhd$ is a binary relation on X defined by $x \rhd y$ iff there exist a $k \in \mathbb{N}$ and $z_1, \ldots, z_k$ in X such that (i) $y \, \mathbf{R}_C \, z_1 \, \mathbf{R}_C \cdots \mathbf{R}_C \, z_k \, \mathbf{R}_C \, x$, (ii) $y \in C\{y, z_1, \ldots, z_k\}$, (iii) $\{x, z_1, \ldots, z_k\} \subseteq C(S)$ for some $S \in \mathbf{k}(X)$ with $y \in S$, and (iv) $y \notin C\{x, y, z_1, \ldots, z_k\}$. The proof is given in the Online Appendix of this paper, available in the websites of either of the authors.

[27] Full continuity of $(\succsim, \mathbf{R})$ is not needed for this result; it is enough to take $\succsim$ to be continuous.

6 Conclusion

In this paper, we proposed a new model to describe the preferences of an economic agent on an arbitrarily given set X of choice prospects. The classical approach is to use a complete preorder, which is typically referred to as a *preference relation*, on X for this purpose. Instead, we suggested the use of two binary relations on X. The first of these, denoted as $\succsim$, aims to capture those rankings of the agent that are (subjectively) "obvious/easy." (This relation is not observable.) As it is hard to imagine that cyclical choice patterns would arise from the "easy" pairwise choice problems, we assume that $\succsim$ is reflexive and transitive, but it need not be complete (because some pairwise choice problems may well be deemed "hard" by the agent). The second binary relation, denoted as $\mathbf{R}$, arises from what we observe the agent choose in the context of all pairwise choice problems. (This relation is observable.) As these include the "hard" ones as well, this relation may exhibit cycles, so it is allowed to be nontransitive, while, naturally, we assume that it is complete. Finally, we posit that $\succsim$ and $\mathbf{R}$ are consistent with each other (as they arise from the preferences of the same agent) in the sense that (i) $\mathbf{R}$ is an extension of $\succsim$, and (ii) $\mathbf{R}$ is transitive with respect to $\succsim$. This way we arrive at what we dubbed here as a *preference structure*. We have showed above that many preference models (where the economic agent may be a group of individuals) are captured by such structures. Among these are the models of incomplete preferences, preferences with imperfect ability of discrimination, regret preferences, and preferences completed by the recommendations of a consultant.

As the main goal of this paper, we have developed a model of choice behavior that arises from preference structures by using the notion of top-cycles. This led to a rich theory of choice which generalizes the classical rational choice theory. The explanatory power of this alternative choice theory is obviously superior to the classical theory. It also has a good deal of predictive power (although, of course, less than the classical theory), for it is a menu-independent model that satisfies, say, the Condorcet Criterion. Moreover, this theory has appealing existence and uniqueness properties, paralleling those of the standard rational choice theory. Indeed, one of our main results establishes the nonempty-valuedness of choice correspondences that are induced by preference structures, and another one identifies the largest preference structure that rationalizes a choice correspondence that is known to be rationalizable by some such structure.

We would like to think of the present paper as a beginning of an extensive research project with numerous avenues to be explored. It would be interesting to revisit the classical consumer theory, this time using preference structures instead of preference relations. Similarly, and even more interestingly, one should investigate how (ordinal) game theory would look like when we model the preferences of the players through preference structures. Then, one should certainly see how the classical theories of decision-making under risk and uncertainty would adapt to preference structures. This would, in turn, open up a whole new set of potential applications. Similarly, it should be interesting to see how one may model time preferences through preference structures, and then revisit the theory of optimal saving. These, and numerous other avenues that remain to be explored, will eventually determine if the notion of preference structures is indeed a useful construct for decision theory at large.

APPENDIX: Proofs

This appendix contains the proofs of the results that were omitted in the body of the text.

Proof of Proposition 3.1

Let X be a nonempty set, and $(\succsim, \mathbf{R})$ a weak preference structure on X. That $\mathbf{R}_{\succsim}$ is a completion of $\succsim$ follows readily from the definition of $\mathbf{R}_{\succsim}$, so we need only to prove that $\mathbf{R}$ is $\succsim$-transitive. To this end, take any x, y and z in X such that $x\ \mathbf{R}_{\succsim}\ y \succsim z$. Notice that $z \succ x$ cannot hold, because otherwise $y \succ x$ (by transitivity of $\succsim$), and hence $y\ (\mathbf{R}_{\succsim})^{>}\ x$ (because $\mathbf{R}_{\succsim}$ is an extension of $\succsim$), a contradiction. Thus: Either $x \succsim z$ or $(x, z) \in \mathrm{Inc}(\succsim)$. In the former case, we have $x\ \mathbf{R}_{\succsim}\ z$ by definition of $\mathbf{R}_{\succsim}$, so we are done. Similarly, if $x \succsim y$, then $x\ \mathbf{R}_{\succsim}\ z$ because $\succsim$ is transitive and $\succsim\ \subseteq \mathbf{R}_{\succsim}$. So, assume that $(x, z) \in \mathrm{Inc}(\succsim)$ and $x \succsim y$ is false. Since $x\ \mathbf{R}_{\succsim}\ y$, the latter statement and the definition of $\mathbf{R}_{\succsim}$ imply that $x\ \mathbf{R}\ y$. Then, $x\ \mathbf{R}\ y \succsim z$, and hence $x\ \mathbf{R}\ z$ by $\succsim$-transitivity of $\mathbf{R}$. It follows that $(x, z) \in \mathrm{Inc}(\succsim)$ and $x\ \mathbf{R}\ z$, that is, $x\ \mathbf{R}_{\succsim}\ z$, as we sought. As we can similarly show that $x \succsim y\ \mathbf{R}_{\succsim}\ z$ implies $x\ \mathbf{R}_{\succsim}\ z$, we conclude that $(\succsim, \mathbf{R}_{\succsim})$ is a preference structure on X.

Proof of Corollary 3.2

Let X be a nonempty set, and $(\succsim, \mathbf{R})$ a transitive weak preference structure on X. In view of Proposition 3.1, we only need to prove that $\mathbf{R}_{\succsim}$ is quasitransitive. We will in fact prove something stronger than this below.

First, recall that a binary relation $\mathbf{S}$ is said to be **Suzumura consistent** if $x\ \mathrm{tran}(\mathbf{S})\ y$ implies not $y\ \mathbf{S}^{>}\ x$ for every $x, y \in X$. Second, note that it is easy to verify that if $(\succsim, \mathbf{S})$ is a preference structure such that $\mathrm{Inc}(\succsim) \cap \mathbf{S}$ is Suzumura consistent, then $\mathbf{S}$ is quasitransitive. Thus, Proposition 3.1 will be proved if we can show that $\mathrm{Inc}(\succsim) \cap \mathbf{R}_{\succsim}$ is Suzumura consistent. To this end, put $\mathbf{T} := \mathrm{Inc}(\succsim) \cap \mathrm{tran}(\mathbf{R}_{\succsim})$, and note that

$$\mathbf{T} = \mathrm{Inc}(\succsim) \cap \mathbf{R}_{\succsim}. \tag{7}$$

(Indeed, for any $(x, y) \in \mathbf{T}$, there exist finitely many $x_1, ..., x_k \in X$ such that $x\ \mathbf{R}_{\succsim}\ x_1\ \mathbf{R}_{\succsim} \cdots \mathbf{R}_{\succsim}\ x_k\ \mathbf{R}_{\succsim}\ y$. Since $\mathbf{R}_{\succsim} \subseteq\ \succsim \cup\ \mathbf{R} \subseteq \mathbf{R}$, we then have $x\ \mathbf{R}\ x_1 \mathbf{R} \cdots \mathbf{R}\ x_k\ \mathbf{R}\ y$, so given that $\mathbf{R}$ is transitive, we find $x\ \mathbf{R}\ y$. As $(x, y) \in \mathrm{Inc}(\succsim)$, this means $x\ \mathbf{R}_{\succsim}\ y$. Thus, $\mathbf{T} \subseteq \mathrm{Inc}(\succsim) \cap \mathbf{R}_{\succsim}$, while the converse containment is trivially true.) Now, to derive a contradiction, suppose $\mathrm{Inc}(\succsim) \cap \mathbf{R}_{\succsim}$ is not Suzumura consistent. Then, by (7), $\mathbf{T}$ is not Suzumura consistent, so there exists a $k \in \mathbb{N}$ and $x_1, \ldots, x_k$ in X such that $x_1\, \mathbf{T} \cdots \mathbf{T}\, x_k\, \mathbf{T}\, x_1$ with at least one of these $\mathbf{T}$ holding strictly. By relabeling if necessary, we may assume that $x_k\, \mathbf{T}^{>}\, x_1$. Then, $x_1\, \mathbf{R}_{\succsim} \cdots \mathbf{R}_{\succsim}\, x_k\, \mathbf{T}^{>}\, x_1$, that is, $x_1\, \mathrm{tran}(\mathbf{R}_{\succsim})\, x_k\, (\mathbf{R}_{\succsim})^{>}\, x_1$. Since x_1 and x_k are not $\succsim$-comparable, it follows that $x_1\, \mathbf{T}\, x_k\, \mathbf{T}^{>}\, x_1$, a contradiction.

Proof of Theorem 3.3

The proof of the "if" part of the assertion is straightforward, so we focus only on its "only if" part. Let $(\succsim, \mathbf{R})$ be a preference structure on X. (The proof for a weak preference structure is given analogously.) Put $\mathbf{T} := \mathbf{R} \setminus \succsim$. We may assume that $\mathbf{T}$ is nonempty, for otherwise there is nothing to prove.

Claim. For every $(x, y) \in \mathbf{T}$, there is a preorder $\succsim_{(x,y)}$ on X such that (i) $\succsim_{(x,y)}$ extends $\succsim$, (ii) $\succsim_{(x,y)} \subseteq \mathbf{R}$, and (iii) $x \succsim_{(x,y)} y$.[28]

Proof of Claim. Fix any $(x, y) \in \mathbf{T}$, and define

$$\succsim_{(x,y)} \quad := \succsim \cup\, (x^{\uparrow, \succsim} \times y^{\downarrow, \succsim}).$$

That $\succsim_{(x,y)}$ is a preorder with $x \succsim_{(x,y)} y$ is verified routinely. To prove (ii), take any $a, b \in X$ with $a \succsim_{(x,y)} b$. If (a, b) does not belong to $\mathbf{R}$, then it does not belong to $\succsim$ either (because $\mathbf{R}$ is a superrelation of $\succsim$). In that case, then, (a, b) belongs to $x^{\uparrow, \succsim} \times y^{\downarrow, \succsim}$, so we have $a \succsim x\ \mathbf{R}\ y \succsim b$, which, by $\succsim$-transitivity of $\mathbf{R}$, implies $a\ \mathbf{R}\ b$, a contradiction. We thus conclude that $\succsim_{(x,y)} \subseteq \mathbf{R}$. It remains to check that $\succsim_{(x,y)}$

[28] When $(\succsim, \mathbf{R})$ is a weak preference structure, we verify a weaker property of (i), namely, that $\succsim_{(x,y)}$ includes $\succsim$, which trivially follows by the construction of $\succsim_{(x,y)}$.

extends $\succsim$. Obviously, $\succsim$ is a subrelation of $\succsim_{(x,y)}$. To complete the proof of the claim, then, take any $a, b \in X$ with $a \succ b$. To derive a contradiction, suppose we have $b \succsim_{(x,y)} a$. Then, by definition of $\succsim_{(x,y)}$, (b, a) must belong to $x^{\uparrow,\succsim} \times y^{\downarrow,\succsim}$, and hence, $y \succsim a \succ b \succsim x$. As $\succsim$ is transitive, then, $y \succ x$, and this implies $y \; \mathbf{R}^{>} \; x$ (because $\succ$ is a subrelation of $\mathbf{R}^{>}$), but this contradicts the fact that $(x, y) \in \mathbf{R}$. $\|$

For each $(x, y) \in \mathbf{T}$, let $\succsim_{(x,y)}$ be a preorder on X that satisfies the conditions of the claim above, and put

$$\mathcal{P} := \{\succsim_{(x,y)} : (x, y) \in \mathbf{T}\} \cup \{\succsim\}.$$

Then, every element of $\mathcal{P}$ is a preorder on X that extends $\succsim$. As $\succsim$ is a subrelation of $\succsim_{(x,y)}$, and $\succsim_{(x,y)}$ is a subrelation of $\mathbf{R}$, for each $(x, y) \in \mathbf{T}$, it is also plain that $\succsim = \bigcap \mathcal{P}$ and $\bigcup \mathcal{P} \subseteq \mathbf{R}$. On the other hand, if $x \; \mathbf{R} \; y$, then either $x \succsim y$ or $(x, y) \in \mathbf{T}$. In the former case, we obviously have $(x, y) \in \bigcup \mathcal{P}$. In the latter case, $x \succsim_{(x,y)} y$, and we again find $(x, y) \in \bigcup \mathcal{P}$. Conclusion: $\bigcup \mathcal{P} = \mathbf{R}$. Finally, as $\mathbf{R}$ is total, this finding shows that $\bigcup \mathcal{P}$ is total, and our proof is complete.

Proof of Corollary 3.4

For the "if" part of the claim, observe that the hypothesis implies $\mathbf{R}$ is a complete preorder which extends (includes) $\succsim$. Obviously, $\succsim$ is a preorder as it is the intersection of the collection $\mathcal{P}$ of preorders. This concludes that $(\succsim, \mathbf{R})$ is a transitive (weak) preference structure on X. For the "only if" part, we readily obtain the required conditions by setting $\mathcal{P} = \{\succsim, \mathbf{R}\}$.

Proof of Proposition 4.1

We will use the following preliminary result to streamline the argument.

Lemma A.1. *Let S be a nonempty set, and $\mathbf{R}$ a complete binary relation on X such that $\bigcirc(S, \mathbf{R}) \neq \varnothing$. Then, $x \; \mathrm{tran}(\mathbf{R}|_S) \; y$ for every $x, y \in \bigcirc(S, \mathbf{R})$.*

Proof. Suppose the assertion is false. By completeness of $\mathbf{R}$, then, there is a $y \in \bigcirc(S, \mathbf{R})$ such that $A := \{x \in \bigcirc(S, \mathbf{R}) : x \; \mathrm{tran}(\mathbf{R}|_S)^{>} \; y\}$ is nonempty. Then, $A \; \mathrm{tran}(\mathbf{R}|_S)^{>} \; z$, which implies $A \; \mathbf{R}|_S^{>} \; z$ (because $\mathbf{R}$ is complete), for every $z \in \bigcirc(S, \mathbf{R}) \backslash A$. But then A is a proper subset of $\bigcirc(S, \mathbf{R})$ which is an $\mathbf{R}$-highset in S, which contradicts $\bigcirc(S, \mathbf{R})$ being the smallest such set. ■

We now turn to the proof of Proposition 4.1. If $\bigcirc(S, \mathbf{R}) \neq \varnothing$, then Lemma A.1, and the fact that $\bigcirc(S, \mathbf{R})$ is an $\mathbf{R}$-highset in S, readily entail that any one element of $\bigcirc(S, \mathbf{R})$ is a maximum element in S with respect to $\mathrm{tran}(\mathbf{R}|_S)$. In other words, nonemptiness of $\bigcirc(S, \mathbf{R})$ entails that $\max(S,\mathrm{tran}(\mathbf{R}|_S))$ is nonempty. Consequently, it is enough to prove the desired equation under the hypothesis that $\max(S,\mathrm{tran}(\mathbf{R}|_S)) \neq \varnothing$. If x belongs to $\max(S,\mathrm{tran}(\mathbf{R}|_S))$ and y is an element of S that does not, then $y \; \mathbf{R} \; x$ cannot hold, because otherwise, $y \; \mathbf{R} \; x \; \mathrm{tran}(\mathbf{R}|_S) \; S$, and hence, $y \; \mathrm{tran}(\mathbf{R}|_S) \; S$, which means $y \in \max(S,\mathrm{tran}(\mathbf{R}|_S))$, a contradiction. As $\mathbf{R}$ is complete and $\max(S,\mathrm{tran}(\mathbf{R}|_S))$ is nonempty, therefore, we conclude that $\max(S,\mathrm{tran}(\mathbf{R}|_S))$ is an $\mathbf{R}$-highset in S. To derive a contradiction, suppose there is an $\mathbf{R}$-highset in S, say, B, which is a proper subset of $\max(S,\mathrm{tran}(\mathbf{R}|_S))$. Take any x in $\max(S,\mathrm{tran}(\mathbf{R}|_S))$ which does not belong to B, and fix an arbitrary y in B. As $x \; \mathrm{tran}(\mathbf{R}|_S) \; y$, there exist finitely many $a_1, ..., a_k \in S$ such that $x \; \mathbf{R} \; a_1 \; \mathbf{R} \cdots \mathbf{R} \; a_k \; \mathbf{R} \; y$. Then, since B is an $\mathbf{R}$-highset in S that contains y, it must also contain a_k. Continuing inductively with this argument, we see that each a_i, and in fact, x must belong to B, a contradiction. This completes our proof.

In the remaining part of this appendix, we adopt the following two conventions:

Notational Convention: Where a preorder $\succsim$ on a nonempty set X is given (and understood from the context),

$$M(A) := \mathbf{MAX}(A, \succsim) \qquad \text{for any nonempty } A \subseteq X.$$

Notational Convention: For any nonnegative integer k, we put

$$[k] := \{0, ..., k\}.$$

Proof of Theorem 4.3

We need the following fact for the main part of the argument.

Lemma A.2. *Let $\succsim$ be a continuous preorder on X. Then, for every $S \in \mathbf{k}(X)$ and $x \in S \backslash M(S)$, there exists a $y \in M(S)$ with $y \succ x$.*

Proof. Take any $S \in \mathbf{k}(X)$ and $x \in S \backslash M(S)$, and put $T := \{y \in S : y \succsim x\}$. By upper semicontinuity of $\succsim \cap (S \times S)$, T is a closed subset of S. Since S is compact, therefore, T is a compact set in X. Then, by means of a well-known theorem of order-theory, we have $\mathbf{MAX}(T) \neq \varnothing$.[29] Pick any y in this set. Notice that any $z \in S$ with $z \succsim y$ must belong to T (by transitivity of $\succsim$). It follows that y is $\succsim$-maximal in S as well. And, obviously, $y \succsim x$. Besides, since x is not $\succsim$-maximal in S, we have $y \succ x$. ■

We now turn to the proof of Theorem 4.3. Let $(\succsim, \mathbf{R})$ be a continuous preference structure on X. Take any S in $\mathbf{k}(X)$, and note that $M(S) \neq \varnothing$ (Lemma A.2). If there is an $\mathbf{R}$-maximum element in $M(S)$, then, obviously, this element is $\mathrm{tran}(\mathbf{R}|_{M(S)})$-maximum in $M(S)$, and hence it belongs to $\bigcirc(M(S), \mathbf{R})$. Assume, then, there is no $\mathbf{R}$-maximum in $M(S)$. This means that for every $x \in M(S)$, there is a $y \in M(S)$ with $y \ \mathbf{R}^{>} \ x$. Moreover, take an arbitrary $x \in S \backslash M(S)$, and observe that, by Lemma A.2, there exists a $z \in M(S)$ with $z \succ x$. If this x is such that $x \ \mathbf{R} \ y$ for all $y \in M(S)$, then we also have $z \ \mathbf{R} \ y$ for all $y \in M(S)$ by $\succsim$-transitivity of $\mathbf{R}$, which contradicts the hypothesis that there is no $\mathbf{R}$-maximum in $M(S)$. Therefore, $y \ \mathbf{R}^{>} \ x$ for some $y \in M(S)$. Conclusion: For every $x \in S$, there is a $y \in M(S)$ with $y \ \mathbf{R}^{>} \ x$. It follows that $\{y^{\downarrow\downarrow} : y \in M(S)\}$ is an open cover of S, where $y^{\downarrow\downarrow} := \{x \in S : y \ \mathbf{R}^{>} \ x\}$. Since S is compact, then, there is a finite subset T of $M(S)$ such that $\{y^{\downarrow\downarrow} : y \in T\}$ covers S. As T is finite, there is a $\mathrm{tran}(\mathbf{R}|_{M(S)})$-maximum, say, x^*, in T. But for any $x \in M(S)$, there is a $y \in T$ with $y \ \mathbf{R}^{>} \ x$ (since $\{y^{\downarrow\downarrow} : y \in T\}$ covers S), and hence $x^* \ \mathrm{tran}(\mathbf{R}|_{M(S)}) \ y \ \mathbf{R}^{>} \ x$, that is, $x^* \ \mathrm{tran}(\mathbf{R}|_{M(S)}) \ x$. It follows that x^* is a $\mathrm{tran}(\mathbf{R}|_{M(S)})$-maximum in $M(S)$. By Proposition 4.1, therefore, $x^* \in \bigcirc(M(S), \mathbf{R})$. This completes the proof of Theorem 4.3.

The following result shows that, in the context of a preference structure $(\succsim, \mathbf{R})$, the asymmetric part of $\mathbf{R}$ is transitive relative to the asymmetric part of $\succsim$. We will use this fact at several points below.

Lemma A.3. *Let $(\succsim, \mathbf{R})$ be a weak preference structure on a nonempty set X. Then,*

$$x \succsim y \ \mathbf{R}^{>} \ z \ (\text{or } x \ \mathbf{R}^{>} \ y \succsim z) \quad \text{implies} \quad x \ \mathbf{R}^{>} \ z$$

for every $x, y, z \in X$.

Proof. Take any $x, y, z \in X$ with $x \succsim y \ \mathbf{R}^{>} \ z$ but assume that $x \ \mathbf{R}^{>} \ z$ is false. As $\mathbf{R}$ is complete, we then have $z \ \mathbf{R} \ x$. So, $z \ \mathbf{R} \ x \succsim y$ and we find $z \ \mathbf{R} \ y$ contradicting $y \ \mathbf{R}^{>} \ z$. The analogous argument shows that $x \ \mathbf{R}^{>} \ y \succsim z$ implies $x \ \mathbf{R}^{>} \ z$ as well. ■

Proof of Proposition 4.8

Take any $S \in \mathfrak{X}$, and any $x, z \in X$ with $x \succsim z \in C(S)$. We put $T := S \cup \{x\}$; our aim is to show that $x \in C(T)$. Assume first that $x \sim z$ (where $\sim$ is the symmetric part of $\succsim$). In this case, $M(S) \cup \{x\} = M(T)$. So, in view of Proposition 4.1, $z \in C(S)$ implies that $z \ \mathrm{tran}(\mathbf{R}|_{M(S)}) \ M(S)$ while $x \ \mathbf{R}^{=} \ z$ (because $\mathbf{R}$ contains $\succsim$). It follows that $x \ \mathrm{tran}(\mathbf{R}|_{M(T)}) \ M(T)$, so, again by Proposition 4.1, $x \in C(T)$.

Assume now that $x \succ z$. In this case x belongs to $M(T)$, but z does not. To derive a contradiction, let us suppose that x does not belong to $C(T)$. By Proposition 4.1, then, there must exist a $y \in M(T) \backslash \{x\}$ such that

$$y \ \mathrm{tran}(\mathbf{R}|_{M(T)})^{>} \ x. \tag{8}$$

[29] The earliest reference for this result seems to be Wallace (1945).

Now, since $z \in C(S)$ and $y \in M(T)\backslash\{x\}$, and hence $y \in M(S)$, we have z tran$(\mathbf{R}|_{M(S)})$ y, so there is a positive integer k and $w_0, ..., w_k \in M(S)$ such that

$$z = w_0 \mathbf{R} w_1 \mathbf{R} \cdots \mathbf{R} w_k = y.$$

Put $\ell := \max\{i \in [k] : x \succ w_i\}$. (This number is well-defined because $x \succ w_0$.) By (8), and because $\mathbf{R}$ contains $\succsim$, we cannot have $x \succsim w_k$, and hence $\ell \in [k-1]$. But then $w_{\ell+1}, ..., w_k \in M(T)$, and we have

$$x \succ w_\ell \mathbf{R} w_{\ell+1} \mathbf{R} \cdots \mathbf{R} w_k = y,$$

so, by $\succsim$-transitivity of $\mathbf{R}$, we find

$$x \mathbf{R} w_{\ell+1} \mathbf{R} \cdots \mathbf{R} w_k = y.$$

This means x tran$(\mathbf{R}|_{M(T)})$ y, contradicting (8).

Proof of Proposition 4.9

Take any finite $S \in \mathfrak{X}$, and any $x, z \in X$ with $x \mathbf{R} z$ and $\{z\} = C(S)$. Put $T := S \cup \{x\}$; we wish to show that $x \in C(T)$. Suppose first that x is not $\succsim$-maximal in T. Then $y \succ x$ for some $y \in T$. Since T is finite and $\succsim$ is transitive, it is without loss of generality to assume that $y \in M(T)$. Since $y \succ x \mathbf{R} z$, we get $y \mathbf{R} z$ by $\succsim$-transitivity of $\mathbf{R}$. As $\{z\}$ is the top-cycle in $M(S)$, and $y \in M(S)$, we must then have $y = z$. But this means $z \succ x$, and hence $z \mathbf{R}^{>} x$, contradiction. Conclusion: $x \in M(T)$.

Now, since $x \mathbf{R} z$, and $\mathbf{R}$ extends $\succsim$, we do not have $z \succ x$. On the other hand, by Proposition 4.8, $x \succsim z$ implies $x \in C(T)$. It remains to consider the case where $(x, z) \in \mathrm{Inc}(\succsim)$. In this case, $z \in M(T)$. Moreover, as $\{z\}$ is the top-cycle in $M(S)$, we have $z \mathbf{R}^{>} y$ for every $y \in M(S)\backslash\{z\}$. But then, $x \mathbf{R} z$ $\mathbf{R}^{>} y$, and hence x tran$(\mathbf{R}|_{M(T)})$ y, for every $y \in M(T)\backslash\{x, z\}$. By Proposition 4.1, then, $x \in C(T)$, and we are done.

Proof of Theorem 5.2

We begin with proving a preliminary result that will be needed in the main body of the proof. This lemma is stated in the setting of Theorem 5.2.

Lemma A.4. *For any $S \in \mathfrak{X}$, $\succsim \in \mathbb{P}(C)$, and any $(x, y) \in S \times X$ with $x \succsim y$,*

$$x \succsim y \quad \textit{implies} \quad C(S \cup \{y\}) \cap S = C(S).$$

Proof. If y is not $\succsim$-maximal in $S \cup \{y\}$, then $M(S \cup \{y\}) = M(S)$, and the claim follows readily from Proposition 4.1. We thus assume that $y \in M(S \cup \{y\})$. In turn, since $x \succsim y$, this implies that $x \in \mathbf{MAX}(S \cup \{y\})$ and $x \sim y$. Consequently,

$$\{x, y\} \subseteq M(S \cup \{y\}) = M(S) \cup \{y\}. \tag{9}$$

Besides, for any $a, b \in M(S)$, we have

$$a \text{ tran}(\mathbf{R}|_{M(S)})\ b \quad \text{iff} \quad a \text{ tran}(\mathbf{R}|_{M(S)\cup\{y\}})\ b, \tag{10}$$

where we denote $\mathbf{R}_C$ by $\mathbf{R}$ to simplify the notation. (The "only if" part of (10) is trivial. Its "if" part follows from the fact that $z \mathbf{R} y$ implies $z \mathbf{R} x$, and $y \mathbf{R} z$ implies $x \mathbf{R} z$, for any $z \in S$ (because $x \sim y$, and $\mathbf{R}$ is $\succsim$-transitive).) Now, there are three cases to consider.

Case 1. $C(S) = \varnothing$. In this case, by Proposition 4.1, there is no tran$(\mathbf{R}|_{M(S)})$-maximum in $M(S)$. It then follows from (10) that there is no tran$(\mathbf{R}|_{M(S)\cup\{y\}})$-maximum in $M(S) \cup \{y\}$. Since $M(S \cup \{y\}) = M(S) \cup \{y\}$, then, Proposition 4.1 entails $C(S \cup \{y\}) \cap S = \varnothing$.

Case 2. $C(S) \neq \varnothing$ and $x \in S\backslash C(S)$. In this case, we have $C(S)$ $\mathbf{R}^{>}$ $x \sim y$. So, by Lemma A.3, $C(S)$ $\mathbf{R}^{>}$ y, that is, $C(S)$ is an $\mathbf{R}$-highset in $M(S) \cup \{y\}$. As $C(S)$ is obviously an $\mathbf{R}$-cycle and (9) holds, we conclude, by Corollary 4.2, that $C(S)$ is the top-cycle in $M(S \cup \{y\})$ with respect to $\mathbf{R}$, that is, $C(S) = C(S \cup \{y\})$.

Case 3. $x \in C(S)$. Again, obviously, $C(S)$ is an $\mathbf{R}$-cycle. Since $x \sim y$ and $\mathbf{R}$ extends $\succsim$, therefore, $C(S) \cup \{y\}$ is an $\mathbf{R}$-cycle as well. Moreover, $y \sim x\ \mathbf{R}^{>}\ M(S) \backslash C(S)$ and hence, $y\ \mathbf{R}^{>}\ M(S) \backslash C(S)$ by Lemma A.3. It follows that $C(S) \cup \{y\}$ is an $\mathbf{R}$-highset in $M(S) \cup \{y\}$. In view of (9) and Corollary 4.2, then, $C(S) \cup \{y\}$ is the top-cycle in $M(S \cup \{y\})$ with respect to $\mathbf{R}$, and hence, $C(S) \cup \{y\} = C(S \cup \{y\})$. ■

We now turn to the proof of Theorem 5.2 in which $\mathcal{P}$ stands for an arbitrarily fixed nonempty subset of $\mathbb{P}(C)$. We define

$$\trianglerighteq_{\mathcal{P}} := \text{tran}\left(\bigcup \mathcal{P}\right),$$

and write $\triangleright_{\mathcal{P}}$ for the asymmetric part of this preorder. (We wish to show that $\trianglerighteq_{\mathcal{P}}$ is the supremum of $\mathcal{P}$ in $\mathbb{P}(C)$ relative to the partial order $\sqsupseteq$.) We organize our main argument in terms of several claims.

Claim 1. $\trianglerighteq_{\mathcal{P}}$ extends any member of $\mathcal{P}$.

Proof of Claim 1. Take any $\succsim$ in $\mathcal{P}$. Obviously, $\trianglerighteq_{\mathcal{P}}$ contains $\succsim$. Next, take any $x, y \in X$ with $x \succ y$. To derive a contradiction, suppose $x \triangleright_{\mathcal{P}} y$ does not hold. Since $x \trianglerighteq_{\mathcal{P}} y$, this means that $y \trianglerighteq_{\mathcal{P}} x$ holds as well. By definition of $\trianglerighteq_{\mathcal{P}}$, then, there exist a $k \in \mathbb{N}$, $\succsim_1, ..., \succsim_k \in \mathbb{P}(C)$ and $z_0, ..., z_k \in X$ such that

$$x \succ y = z_0 \succsim_1 z_1 \succsim_2 \cdots \succsim_k z_k = x.$$

Put $S := \{z_0, ..., z_k\}$. Since $C(S) \neq \varnothing$, there is an $i \in [k]$ such that $z_i \in C(S)$. If $i > 0$, Proposition 4.8 entails that $z_{i-1} \in C(S)$, and continuing inductively, we find that $y = z_0 \in C(S)$. So, in all contingencies, we have $y \in C(S)$. But this is impossible, because y is not $\succsim$-maximal in S, and $(\succsim, \mathbf{R}_C)$ rationalizes C. □

Claim 2. $(\trianglerighteq_{\mathcal{P}}, \mathbf{R}_C)$ is a preference structure on X.

Proof of Claim 2. Let us first show that $\mathbf{R}_C$ extends $\trianglerighteq_{\mathcal{P}}$. Take any $x, y \in X$ with $x \trianglerighteq_{\mathcal{P}} y$. Then, there exist a $k \in \mathbb{N}$, $\succsim_1, ..., \succsim_k \in \mathbb{P}(C)$ and $z_0, ..., z_k \in X$ such that

$$x = z_0 \succsim_1 z_1 \succsim_2 \cdots \succsim_k z_k = y. \tag{11}$$

If $k = 1$, then $x \succsim_1 y$, and hence $x\ \mathbf{R}_C\ y$ (because $\mathbf{R}_C$ is a superrelation of $\succsim_1$). Suppose $k \geq 2$. Then, $z_{k-2} \succsim_{k-1} z_{k-1} \succsim_k z_k$, and hence $z_{k-2} \succsim_{k-1} z_{k-1}\ \mathbf{R}_C\ z_k$, so by $\succsim_{k-1}$-transitivity of $\mathbf{R}_C$, we find

$$x = z_0 \succsim_1 z_1 \succsim_2 \cdots \succsim_{k-2} z_{k-2} \mathbf{R}_C\ z_k = y.$$

If $k = 2$, we are done. Otherwise, we continue this way inductively to obtain $x\ \mathbf{R}_C\ y$ in $k-1$ steps. Conclusion: $x\ \mathbf{R}_C\ y$. Now assume that we in fact had $x \triangleright_{\mathcal{P}} y$ (which implies that at least one of the orderings in (11) holds strictly). Let us show that $x\ \mathbf{R}_C^{=}\ y$ could not hold in this case. Indeed, to derive a contradiction, suppose $x\ \mathbf{R}_C^{=}\ y$ so that $y \in C\{x, y\}$. Then, since $x \succsim_1 z_1$, Lemma A.4 entails that $C\{x, y\} \subseteq C\{x, y, z_1\}$, so $y \in C\{x, y, z_1\}$. If $k \geq 2$, then, since $z_1 \succsim_1 z_2$, Lemma A.4 entails that $y \in C\{x, y, z_1\} \subseteq C\{x, y, z_1, z_2\}$. Continuing this way inductively, we find that $y \in C\{x, y, z_1, ..., z_{k-1}\}$. But then, since $z_{k-1} \succsim_k y$ and $(\succsim_k, \mathbf{R}_C)$ rationalizes C, Proposition 4.8 yields $z_{k-1} \in C\{x, y, z_1, ..., z_{k-1}\}$. Continuing this way inductively, therefore, we find $\{x, y, z_1, ..., z_{k-1}\} = C\{x, y, z_1, ..., z_{k-1}\}$, but this contradicts the fact that at least one of the orderings in (11) holds strictly. Conclusion: $x \triangleright_{\mathcal{P}} y$ implies $x\ \mathbf{R}_C^{>}\ y$.

It remains to establish that $\mathbf{R}_C$ is $\trianglerighteq_{\mathcal{P}}$-transitive. To this end, take any $x, y, z \in X$ with $x \trianglerighteq_{\mathcal{P}} y\ \mathbf{R}_C\ z$. Then, there are $k \in \mathbb{N}$, $\succsim_1, ..., \succsim_k \in \mathbb{P}(C)$ and $w_0, ..., w_k \in X$ such that

$$x = w_0 \succsim_1 w_1 \succsim_2 \cdots \succsim_k w_k = y\ \mathbf{R}_C\ z.$$

So, repeating the induction argument we gave in the previous paragraph, we find $x\ \mathbf{R}_C\ z$. That $x\ \mathbf{R}_C\ y \trianglerighteq_{\mathcal{P}} z$ implies $x\ \mathbf{R}_C\ z$ is similarly proved. □

Claim 3. For any $S \in \mathfrak{X}$ and $\succsim \in \mathcal{P}$,

$$C(S) \subseteq \mathbf{MAX}(S, \trianglerighteq_{\mathcal{P}}) \subseteq \mathbf{MAX}(S, \succsim).$$

Proof of Claim 3. The second containment is an immediate consequence of Claim 1. To establish the first containment, take any x in $C(S)$, and suppose that x is not $\trianglerighteq_{\mathcal{P}}$-maximal in S. Then, there is a $y \in S$ with $y \rhd_{\mathcal{P}} x$, and hence,

$$y = z_0 \succsim_1 z_1 \succsim_2 \cdots \succsim_k z_k = x$$

for some $k \in \mathbb{N}$, $\succsim_1, ..., \succsim_k \in \mathbb{P}(C)$ and $z_0, ..., z_k \in X$, with at least one of these ordering holding strictly. (We have $k > 1$, for otherwise $y \succ_1 x$, that is, x is not $\succsim_1$-maximal in S, contradicting $x \in C(S)$.) Since $y \in S$ and $y \succsim_1 z_1$, Lemma A.4 tells us that

$$C(S \cup \{z_1\}) \cap S = C(S).$$

But then, since $z_1 \in S \cup \{z_1\}$ and $z_1 \succsim_2 z_2$, applying Lemma A.4 again yields

$$C(S \cup \{z_1, z_2\}) \cap (S \cup \{z_1\}) = C(S \cup \{z_1\}).$$

Intersecting both sides of this equation with S, and using the previous equation, then, we find

$$C(S \cup \{z_1, z_2\}) \cap S = C(S).$$

In fact, proceeding this way inductively, we may conclude that

$$C(S \cup \{z_1, ..., z_k\}) \cap S = C(S). \tag{12}$$

In particular, x belongs to $C(S \cup \{z_1, ..., z_k\})$. But then, by Proposition 4.8, z_{k-1} belongs to $C(S \cup \{z_1, ..., z_k\})$ as well. In fact, applying Proposition 4.8 this way inductively, we find that $z_0, ..., z_k \in C(S \cup \{z_1, ..., z_k\})$. But this is impossible, for $z_{i-1} \succ_i z_i$ holds for at least one $i \in \{1, ..., k\}$, so, being not $\succsim_i$-maximal in $S \cup \{z_1, ..., z_k\}$, z_i cannot belong to $C(S \cup \{z_1, ..., z_k\})$. $\square$

Claim 4. $(\trianglerighteq_{\mathcal{P}}, \mathbf{R}_C)$ rationalizes C.

Proof of Claim 4. Take any $S \in \mathfrak{X}$ and note that $C(S)$ is an $\mathbf{R}_C$-cycle in $C(S)$. So, since, by Claim 3, $C(S) \subseteq \mathbf{MAX}(S, \trianglerighteq_{\mathcal{P}})$, it is plain that $C(S)$ is an $\mathbf{R}_C$-cycle in $\mathbf{MAX}(S, \trianglerighteq_{\mathcal{P}})$. Now pick any $\succsim \in \mathbb{P}(C)$. Then, $C(S)$ is an $\mathbf{R}_C$-highset in $\mathbf{MAX}(S, \succsim)$. By Claim 3, therefore, $C(S)$ is an $\mathbf{R}_C$-highset in $\mathbf{MAX}(S, \trianglerighteq_{\mathcal{P}})$ as well. This means that $C(S)$ is the top-cycle in $\mathbf{MAX}(S, \trianglerighteq_{\mathcal{P}})$ with respect to $\mathbf{R}_C$, as we claimed. $\square$

Claims 2 and 4 jointly imply that $\trianglerighteq_{\mathcal{P}}$ belongs to $\mathbb{P}(C)$. It then follows from Claim 1 that $\trianglerighteq_{\mathcal{P}}$ is the supremum of $\mathcal{P}$ in $\mathbb{P}(C)$ relative to $\sqsupseteq$. In view of the arbitrary choice of $\mathcal{P}$ above, we conclude that $\mathbb{P}(C)$ is a complete $\vee$-semilattice relative to this partial order. The proof of Theorem 5.2 is complete.

Proof of Theorem 5.3

Throughout the proof, we will denote $\mathbf{R}_C$ by $\mathbf{R}$ to simplify the notation. (That is, for any x and y in X, we have $\{x\} = C\{x, y\}$ iff $x \ \mathbf{R}^> \ y$, and $\{x, y\} = C\{x, y\}$ iff $x \ \mathbf{R}^= \ y$.) Consequently, $\succ_C \subseteq \mathbf{R}^>$ and $\sim_C \subseteq \mathbf{R}^=$, that is, that $\mathbf{R}$ extends $\succsim_C$. We will use these facts below as a matter of routine. Also, when $\succsim \in \mathbb{P}(C)$, we again write $M(S) = \mathbf{MAX}(S, \succsim)$ for any $S \in \mathfrak{X}$. The following lemmata are stated in the setting of Theorem 5.3.

Lemma A.5. *For any finite $S \in \mathfrak{X}$,*

$$\{x\} = C(S) \quad \textit{implies} \quad \{x\} = C\{x, y\} \textit{ for every } y \in S.$$

Proof. Take any finite $S \in \mathfrak{X}$ with $\{x\} = C(S)$. If $S = \{x\}$, there is nothing to prove, so assume otherwise, and pick any $y \in S \backslash \{x\}$. Take any $\succsim \in \mathbb{P}(C)$, and denote $\mathbf{R}_C$ by $\mathbf{R}$ to simplify the notation. If $y \in M(S)$, then since $\{x\}$ is the top-cycle in $M(S)$ with respect to $\mathbf{R}$, we have $x \ \mathbf{R}^> \ y$. If y is not $\succsim$-maximal in S, then $z \succ y$ holds for some $z \in S$. As S is finite, we may assume that $z \in M(S)$. If $z = x$, then, obviously, $\{x\} = C\{x, y\}$, so assume that $z \neq x$. Then, since $\{x\} = \bigcirc(M(S), \mathbf{R})$, we have $x \ \mathbf{R}^> \ z \succ y$, so, by Lemma A.3, we again find $x \ \mathbf{R}^> \ y$. Since $\{x\} = C\{x, y\}$ iff $x \ \mathbf{R}^> \ y$, we are done. ■

Lemma A.6. *For any $S \in \mathfrak{X}$ and $x \in S$,*

$$x \succ_C y \quad \text{implies} \quad C(S \cup \{y\}) = C(S).$$

Proof. Take any $S \in \mathfrak{X}$. Let x and y be two elements of X with $x \succ_C y$. Then, $x\ \mathbf{R}^{>}\ y$, so $y \succsim x$ cannot hold (because $\mathbf{R}$ extends $\succsim$). Besides, if there is a $z \in S$ with $z \succ y$, then $C(S \cup \{y\}) \subseteq S$, so, by Lemma A.4, $C(S \cup \{y\}) = C(S)$, and we are done. It remains to consider the case where $(x, y) \in \text{Inc}(\succsim)$ and $y \in M(S \cup \{y\})$.

Since $x \succ_C y$ implies that y does not belong to $C(S \cup \{y\})$, we have

$$C(S \cup \{y\}) \subseteq M(S \cup \{y\}) \cap S \subseteq M(S).$$

Now, we claim that $C(S \cup \{y\})$ is an $\mathbf{R}$-highset in $M(S)$ To see this, take any $\succsim$-maximal z in S that does not belong to $C(S \cup \{y\})$. If $z \in M(S \cup \{y\})$, then we clearly have $C(S \cup \{y\})\ \mathbf{R}^{>}\ z$ (because $C(S \cup \{y\})$ is an $\mathbf{R}$-highset in $M(S \cup \{y\})$. If $z \notin M(S \cup \{y\})$, then, since $y \in M(S \cup \{y\}) \backslash C(S \cup \{y\})$, we have $C(S \cup \{y\})\ \mathbf{R}^{>}\ y \succ z$, which implies $C(S \cup \{y\})\ \mathbf{R}^{>}\ z$ by Lemma A.3. So, $C(S \cup \{y\})$ is an $\mathbf{R}$-highset in $M(S)$. As $C(S \cup \{y\})$ is obviously an $\mathbf{R}$-cycle, $C(S \cup \{y\}) = C(S)$ by Corollary 4.2. ■

We now turn to the proof of Theorem 5.3.

Claim 1. $\mathbf{R}$ is $\succsim_C$-transitive.

Proof of Claim 1. Let us first show that $\mathbf{R}$ is $\succ_C$-transitive. Take any $x, y, z \in X$ with $x\ \mathbf{R}\ y \succ_C z$. If $x\ \mathbf{R}\ z$ is false, then $z\ \mathbf{R}^{>}\ x$ (because $\mathbf{R}$ is complete). Take any $\succsim$ in $\mathbb{P}(C)$. Clearly, $y \succ x$ cannot hold (because $\succsim \subseteq \mathbf{R}$). If, on the other hand, $x \succ y$, then $\{x\} = C\{x, y, z\}$, and by Lemma A.5, this implies $x\ \mathbf{R}^{>}\ z$, a contradiction. Thus: $(x, y) \in \text{Inc}(\succ)$. Similarly, $x \succ z$ cannot hold (because $\succsim \subseteq \mathbf{R}$), and if $z \succ x$, then $\{y\} = C\{x, y, z\}$, and by Lemma A.5, this implies $y\ \mathbf{R}^{>}\ x$, a contradiction. Thus: $(x, y) \in \text{Inc}(\succ)$. Finally, note that $z \succ y$ cannot hold (because $\succ \subseteq \succ_C$), and if $y \succ z$, then Lemma A.3 implies $y\ \mathbf{R}^{>}\ x$, a contradiction. Thus: $(y, z) \in \text{Inc}(\succ)$. Conclusion: $\mathbf{MAX}(\{x, y, z\}, \succsim) = \{x, y, z\}$. Then, by Proposition 4.1, $\{x, y, z\} = C\{x, y, z\}$, but this contradicts $y \succ_C z$. Thus: $\mathbf{R} \circ \succ_C \subseteq \mathbf{R}$. One can similarly prove that $\succ_C \circ \mathbf{R} \subseteq \mathbf{R}$.

We next show that $\mathbf{R}$ is $\sim_C$-transitive. Take any $x, y, z \in X$ with $x\,\mathbf{R}\,y \sim_C z$. The second part of this statement entails that $x \in C(\{x\} \cup \{y\})$ iff $x \in C(\{x\} \cup \{z\})$. But $x \in C\{x, y\}$ (because $x\ \mathbf{R}\ y$), so we find that $x \in C\{x, z\}$, that is, $x\ \mathbf{R}\ z$. We thus conclude that $\mathbf{R} \circ \sim_C \subseteq \mathbf{R}$. One can similarly prove that $\sim_C \circ \mathbf{R} \subseteq \mathbf{R}$. □

Claim 2. $\succsim_C$ is transitive.

Proof of Claim 2. Observe that it is sufficient to verify (i) $\sim_C$ is transitive, (ii) $\succ_C$ is transitive, and (iii) $\succ_C$ is $\sim_C$-transitive. The construction of $\sim_C$ readily implies (i). For (ii), suppose that $x \succ_C y \succ_C z$ but $x \succ_C z$ does not hold for some $x, y, z \in X$. Then, there exists an $S \in \mathfrak{X}$ with $x \in S$ and $z \in C(S)$. As $x \succ_C y$, Lemma A.6 implies that $z \in C(S) = C(S \cup \{y\})$. But this is a contradiction since $y \succ_C z$. Thus, $x \succ_C y \succ_C z$ must imply $x \succ_C z$. For (iii), take any $x, y, z \in X$ such that $x \succ_C y \sim_C z$. Let S be an arbitrary element of $\mathfrak{X}$ with $x \in S$. (We wish to show that z does not belong to $C(S)$.) If z does not belong to S, there is nothing to prove, so suppose $z \in S$. As $x \succ_C y$ and $x \in S$, we have $y \notin C(S \cup \{y\})$. Thus, since $y \sim_C z$, we have $z \notin C(S \cup \{z\}) = C(S)$. An analogous argument shows that $x \sim_C y \succ_C z$ implies $x \succ_C z$ as well. □

Claim 3. For any $\succsim \in \mathbb{P}(C)$, $\succsim_C$ extends $\succsim$.

Proof of Claim 3. Let $\succsim \in \mathbb{P}(C)$. If $x \succ y$ for some $x, y \in X$, then $y \notin M(S)$ and thus $y \notin C(S)$ for all $S \in \mathfrak{X}$ with $x \in S$. So, $\succ \subseteq \succ_C$. In the rest of the proof, we prove that $\sim \subseteq \sim_C$. Take any $x, y \in X$ with $x \sim y$. Let S be an arbitrarily fixed element of $\mathfrak{X}$, and put $T_x := M(S \cup \{x\})$ and $T_y := M(S \cup \{y\})$. Since $\succsim$ is a preorder, it is readily checked that $x \sim y$ implies $T := T_x \cap S = T_y \cap S$.

Now assume $x \in C(S \cup \{x\})$. Then, $x \in M(S \cup \{x\})$, and hence, $y \in M(S \cup \{y\})$ (because $\succsim$ is a preorder and $x \sim y$), that is, $y \in T_y$. Moreover, by Proposition 4.1, $x\ \text{tran}(\mathbf{R}|_{T_x})\ T_x$. So, if $z \in T \subseteq T_x$, there is a positive integer k such that $x\ \mathbf{R}\ w_0\ \mathbf{R} \cdots \mathbf{R}\ w_k = z$ for some $w_0, ..., w_k \in T_x$. Here, we can in fact assume that $w_0, ..., w_k \in T$ without loss of generality. (For, otherwise, $w_i = x$ for some $i \in [k]$.

Then, set $l := \max\{i \in [k] : w_i = x\}$, and we have $x \mathbf{R} w_{l+1} \mathbf{R} \cdots \mathbf{R} w_k = z$ with $w_i \in T$ for all $i = l+1, \dots, k$.) Since $y \sim x$ and $\mathbf{R}$ is $\succsim$-transitive, this implies $y \mathbf{R} w_0 \mathbf{R} \cdots \mathbf{R} w_k = z$. Thus, $y \operatorname{tran}(\mathbf{R}|_{T_y}) T$. If $z \in T_y \backslash T$, then $z = y$, and we obviously have $y \operatorname{tran}(\mathbf{R}|_{T_y}) z$. Conclusion: $y \operatorname{tran}(\mathbf{R}|_{T_y}) T_y$. By Proposition 4.1, this yields $y \in C(S \cup \{y\})$, as we sought. By symmetry, therefore, we conclude: $x \in C(S \cup \{x\})$ iff $y \in C(S \cup \{y\})$.

Next, take any z in S with $z \in C(S \cup \{x\})$. Then, $z \in M(S \cup \{x\})$, and hence, $z \in M(S \cup \{y\})$ (because $\succsim$ is a preorder and $x \sim y$), that is, $z \in T_y$. Moreover, by Proposition 4.1, $z \operatorname{tran}(\mathbf{R}|_{T_x}) T_x$. Now, take any $w \in T_y$. Then, we can show that there is a positive integer k with

$$z \mathbf{R} w_0 \mathbf{R} w_1 \mathbf{R} \cdots \mathbf{R} w_k \mathbf{R} w \quad \text{ for some } w_0, \dots, w_k \in T_x. \tag{13}$$

(Indeed, if $w \in T \subseteq T_x$, then $z \operatorname{tran}(\mathbf{R}|_{T_x}) w$, and (13) follows at once. If $w \in T_y \setminus T$, then $y = w \in T_y$, which implies $x \in T_x$ and hence $z \operatorname{tran}(\mathbf{R}|_{T_x}) x \sim y$. So, there is a positive interger k such that $z \mathbf{R} w_0 \mathbf{R} w_1 \mathbf{R} \cdots \mathbf{R} w_k \mathbf{R} x \sim y$ for some $w_0, \dots, w_k \in T_x$. This again implies (13) by $\succsim$-transitivity of $\mathbf{R}$.) For the sequence $w_0, ..., w_k$ in (13), define

$$w_i' := \begin{cases} w_i, & \text{if } w_i \neq x \\ y, & \text{if } w_i = x, \end{cases}$$

for each $i \in [k]$, and note that $z \mathbf{R} w_1' \mathbf{R} \cdots \mathbf{R} w_k' \mathbf{R} w$ by $\succsim$-transitivity of $\mathbf{R}$. Since $w_i' \in T_y$ for each $i \in [k]$, this shows that $z \operatorname{tran}(\mathbf{R}|_{T_y}) w$. It then follows from the arbitrary choice of w that $z \operatorname{tran}(\mathbf{R}|_{T_y}) T_y$, that is, $z \in C(S \cup \{y\})$, as we sought. By symmetry, therefore, we conclude: $z \in C(S \cup \{x\})$ iff $z \in C(S \cup \{y\})$ for every $z \in S$. In view of the arbitrariness of S, this establishes that $\sim \subseteq \sim_C$. $\square$

In view of Theorem 5.2, we have $\bigvee \mathbb{P}(C) \in \mathbb{P}(C)$. So, by Claim 3, it follows that $\bigvee \mathbb{P}(C) \subseteq \succsim_C$. As $\mathbf{R}$ extends $\succsim_C$, Claim 1 and Claim 2 imply that $(\succsim_C, \mathbf{R})$ is a preference structure on X. Since $\bigvee \mathbb{P}(C)$ is the largest preorder in $\mathbb{P}(C)$, if $(\succsim_C, \mathbf{R})$ rationalizes C, then $\succsim_C \subseteq \bigvee \mathbb{P}(C)$. The next claim estabilishes this step, hence completing the proof of Theorem 5.3.

Claim 4. $(\succsim_C, \mathbf{R})$ rationalizes C.

Proof of Claim 4. Take any S in $\mathfrak{X}$ and $\succsim \in \mathbb{P}(C)$. If $x \in C(S)$, then $y \succ_C x$ holds for no $y \in S$ by definition of $\succ_C$, implying that $x \in \mathbf{MAX}(S, \succsim_C)$. So, $C(S) \subseteq \mathbf{MAX}(S, \succsim_C)$. In addition, we have $\mathbf{MAX}(S, \succsim_C) \subseteq M(S)$ as $\succsim_C$ extends $\succsim$ by Claim 3. These observations readily imply that $C(S)$ is an $\mathbf{R}$-highset in $\mathbf{MAX}(S, \succsim_C)$. (Indeed, if $x \in C(S)$ and $y \in \mathbf{MAX}(S, \succsim_C) \backslash C(S)$, then $y \in M(C) \backslash C(S)$ and thus $x \mathbf{R}^{>} y$.) As $C(S)$ is obviously an $\mathbf{R}$-cycle, we conclude that $C(S) = \bigcirc(\mathbf{MAX}(S, \succsim_C), \mathbf{R})$ by Corollary 4.2. The proof is complete. $\square$

Proof of Theorem 5.4

In view of Theorems 4.3, 5.2 and 5.3, $\mathbb{P}(C)$ is a complete $\vee$-semilattice such that $\bigvee \mathbb{P}(C) = \succsim_C$. It remains to prove that $\mathbb{P}(C)$ is a complete $\wedge$-semilattice as well. To this end, fix an arbitrary nonempty subset $\mathcal{P}$ of $\mathbb{P}(C)$, and define $\trianglerighteq := \bigcap \mathcal{P}$. We will complete our proof by showing that $\trianglerighteq$ belongs to $\mathbb{P}(C)$. It is readily checked that $\trianglerighteq$ is a preorder on X and $\mathbf{R}$ is a $\trianglerighteq$-transitive extension of $\trianglerighteq$, that is, $(\trianglerighteq, \mathbf{R})$ is a preference structure. Below we prove that $(\trianglerighteq, \mathbf{R})$ rationalizes C.

Claim. Every $\succsim$ in $\mathcal{P}$ extends $\trianglerighteq$.

Proof of Claim. If $x \vartriangleright y$, then $x \succsim y$ for every $\succsim \in \mathcal{P}$ and $x \succ^* y$ for some $\succsim^* \in \mathcal{P}$. If $x \sim y$ holds for some $\succsim \in \mathcal{P}$, then $x \mathbf{R}^{=} y$, while $x \succ^* y$ implies $x \mathbf{R}^{>} y$ (since $\mathbf{R}$ extends both $\succsim$ and $\succsim^*$), a contradiction. Thus, we must have $x \succ y$ for all $\succsim \in \mathcal{P}$. $\square$

Let us now fix an arbitrary S in $\mathbf{k}(X)$. Note that

$$C(S) \subseteq \mathbf{MAX}(S, \succsim) \subseteq \mathbf{MAX}(S, \trianglerighteq)$$

for every $\succsim \in \mathcal{P}$. (Here the first containment follows from the fact that $(\succsim, \mathbf{R})$ rationalizes C, and the second from the Claim above.) Obviously, $C(S)$ is an $\mathbf{R}$-cycle. We claim that it is also an $\mathbf{R}$-highset in $\mathbf{MAX}(S, \trianglerighteq)$. To prove this, take any $x \in C(S)$ and $y \in \mathbf{MAX}(S, \trianglerighteq) \backslash C(S)$, and suppose, to derive a contradiction, that $y \mathbf{R} x$. If $y \in \mathbf{MAX}(S, \succsim)$ for some $\succsim \in \mathcal{P}$, then $x \mathbf{R}^{>} y$ (because $C(S)$ is an $\mathbf{R}$-highset in $\mathbf{MAX}(S, \succsim)$), a contradiction. Assume, then, y is not $\succsim$-maximal in S for any $\succsim \in \mathcal{P}$. Next, fix an arbitrary element $\succsim^*$ of $\mathcal{P}$. Since y is not $\succsim^*$-maximal in S, Lemma A.2 entails that there is a $z \in \mathbf{MAX}(S, \succsim^*)$ with $z \succ^* y$. If z does not belong to $C(S)$, then $x \mathbf{R}^{>} z \succ^* y$, and hence, $x \mathbf{R}^{>} y$ (Lemma A.3), again a contradiction. So, we assume that $z \in C(S)$. Then, as $C(S)$ is an $\mathbf{R}$-cycle, there exist a $k \in \mathbb{N}$ and $w_0, ..., w_k \in C(S)$ such that

$$y \mathbf{R} x = w_0 \mathbf{R} w_1 \mathbf{R} \cdots \mathbf{R} w_k = z. \tag{14}$$

Define $\ell := \min\{i \in [k] : w_i \succ^* y\}$. This number is well-defined, because $w_k = z \succ^* y$. Moreover, $\ell > 0$, because, as $y \mathbf{R} x = w_0$, and $\mathbf{R}$ extends $\succsim^*$, we cannot have $w_0 \succ^* y$. Now, by definition of ℓ, y is $\succsim^*$-maximal in $\{y, w_0, ..., w_{\ell-1}\}$, and hence, since $(\succsim^*, \mathbf{R})$ rationalizes C, (14) implies

$$y \in C\{y, w_0, ..., w_{\ell-1}\}, \tag{15}$$

while, of course,

$$y \notin C\{y, w_0, ..., w_\ell\}. \tag{16}$$

But then, an immediate application of Proposition 4.1 shows that (14), (15) and (16) may jointly hold only if $w_\ell \succ y$ for every $\succsim \in \mathcal{P}$ (because $(\succsim, \mathbf{R})$ rationalizes C for each $\succsim \in \mathcal{P}$). Since $w_\ell \in S$, it follows that y is not $\trianglerighteq$-maximal in S, in contradiction to the choice of y. Conclusion: $C(S)$ is an $\mathbf{R}$-highset in $\mathbf{MAX}(S, \trianglerighteq)$, and hence, $C(S) = \bigcirc(\mathbf{MAX}(S, \trianglerighteq), \mathbf{R})$ by Corollary 4.2. The proof of Theorem 5.4 is now complete.

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