Top-Cycles and Revealed Preference Structures*

Ozgur Evren[†] Hiroki Nishimura[‡] Efe A. Ok[§]

December 20, 2019

Abstract

A *preference structure* is a pair of binary relations $(\succsim, \mathbf{R})$ on a given alternative space such that $\succsim$ is reflexive and transitive, and $\mathbf{R}$ is a completion of $\succsim$ which is transitive with respect to $\succsim$. Here $\succsim$ captures the comparisons that a decision maker is able to make with full confidence, perhaps because $\succsim$ reflects the joint rankings of a committee of experts, or because it agrees with the preferences of all of her potential selves. In turn, $\mathbf{R}$ captures the (observed) preferences of the agent revealed through pairwise choice problems. We define choices induced by a preference structure on a feasible menu S as the top-cycle (relative to $\mathbf{R}$) among all undominated alternatives in S (relative to $\succsim$), and investigate this choice model from the standpoint of revealed preference theory. First, we identify a set of behavioral postulates that characterize those choice correspondences that are rationalized by some preference structure $(\succsim, \mathbf{R})$ with and without the requirement that $\mathbf{R}$ be transitive. (As a by product, we also obtain a simple characterization of the celebrated top-cycle choice rule.) Second, we characterize the set of all preference structures that rationalize a given choice correspondence, and examine the extent to which one's choice behavior "reveals" her preference structure. At several points in the exposition, we demonstrate our findings in the concrete context of decision-making under uncertainty.

JEL Classification: D11, D81.
Keywords: Top-cycle choice rules, preference structures, revealed preference theory, boundedly rational choice, incomplete preferences, nontransitive preferences.

*We thank the participants of the seminars given in UCSD, University of Pennsylvania, Virginia Tech and Washington University in St. Louis for their comments, and Fuad Aleskerov and Gleb Koshevoy for providing us with some historical references on the fundamental axioms of choice theory. The second and third authors gratefully acknowledge the support of the National Science Foundation under the grants NSF-1851610 and NSF-1851639.

[†]New Economic School, Moscow. Email: oevren@nes.ru.
[‡]Department of Economics, University of California Riverside. Email: hiroki.nishimura@ucr.edu.
[§]Department of Economics and the Courant Institute of Mathematical Sciences, New York University. E-mail: efe.ok@nyu.edu.

1 Introduction

Consider a decision maker (the agent) who uses a *collection* $\mathcal{P}$ of complete and transitive preference relations over a given set of choice prospects. For instance, the elements of $\mathcal{P}$ may correspond to the preferences of a set of experts employed by the decision maker. Alternatively, we may think of any one member of $\mathcal{P}$ as the ranking of alternatives according to a certain criterion (as in multi-criteria decision making models). Or, the elements of $\mathcal{P}$ may be describing the preferences of different selves of the agent (as in behavioral multi-selves models). In any one of these scenarios, the ultimate ranking of the alternatives is likely to arise through some form of aggregation of the members of $\mathcal{P}$. For instance, the decision maker may adopt one of the members of $\mathcal{P}$ and ignore the rest of them, rank the alternatives according to a lexicographic order induced by $\mathcal{P}$, or, more realistically, she may utilize some form of a majority voting scheme.

We denote the decision maker's pairwise rankings of the alternatives by a binary relation $\mathbf{R}$. Thus, $x \mathbf{R} y$ means that the agent deems x choosable from the pairwise menu $\{x, y\}$ (because, say, we have observed her choose x from this menu at least once). Given this interpretation, it is plain that $\mathbf{R}$ should be complete, but if, for instance, we wish to capture the majority voting scenario above, we cannot presuppose that it is transitive. More generally, when the members of $\mathcal{P}$ disagree about the ranking of x and y, the problem of choosing from $\{x, y\}$ becomes "hard," and it is known that the presence of such hard problems may yield nontransitive choices.[1]

Whether it is transitive or not, it is natural to posit that $\mathbf{R}$ is consistent with $\mathcal{P}$ in the following sense: If all members of $\mathcal{P}$ rank x above y, then so does $\mathbf{R}$. In other words, where $\succsim$ stands for the intersection of all members of $\mathcal{P}$ – $\succsim$ is thus a transitive, but not necessarily complete, preference relation – it makes sense to assume that $x \succsim y$ implies $x \mathbf{R} y$. In fact, if all members of $\mathcal{P}$ rank x over y weakly, and at least one of them strictly, then it stands to reason that $\mathbf{R}$ ranks x strictly above y. (In the jargon of order theory, this means that $\mathbf{R}$ is an *extension* of $\succsim$.) One may choose to impose further connections between $\succsim$ and $\mathbf{R}$, and indeed, we will work with an additional assumption below, but let us leave this issue for a moment, and turn to modeling the "choices" of the agent from an arbitrarily given finite feasible menu S.

A natural starting point would be to look at the alternatives in S that maximize the agent's revealed preferences, $\mathbf{R}$. But, since $\mathbf{R}$ is not necessarily transitive, there may not be any such alternative in S, leaving us without a prediction.[2] This difficulty is, of course, well-known, and the most prominent remedy offered by the literature on this issue is to consider the transitive closure of the restriction of $\mathbf{R}$ to the menu S. The maximizers of this

[1]While the best known example is the majority voting scenario above, Luce and Raiffa (1957) put this more generally: "... intransitivities often occur when a subject forces choices between inherently incomparable alternatives." Since then, several authors reemphasized the idea that the revealed preferences of an individual may be nontransitive due to the presence of alternatives that are difficult to compare. A formalization of this phenomenon can be found in Mandler (2005), among others. In addition, Costa-Gomes et al. (2019) have recently provided compelling (experimental) evidence to this effect.

[2]Many examples in the literature feature a transitive relation $\mathbf{R}$ that contains $\succsim$ but does not respect its strict part. In such cases, maximizing $\mathbf{R}$ on S leads to a further difficulty: A maximum in S with respect to $\mathbf{R}$ may be strictly dominated in terms of $\succsim$, making it difficult to believe that the agent would ever choose that alternative. We elaborate on this issue in Section 4.3.

complete preorder on S are called the *top-cycle* choices in S with respect to $\mathbf{R}$. Denoting the totality of these choices by $\bigcirc(S, \mathbf{R})$, we are led to set

$$C(S) = \bigcirc(S, \mathbf{R}), \tag{1}$$

where C stands for the choice correspondence of the decision maker. Introduced by Kalai, Pazner and Schmeidler (1976), and Kalai and Schmeidler (1977), this choice model has attracted considerable attention in the literature, especially in the context of social choice problems and tournaments. Nice behavioral characterizations of the top-cycle choice rule are also available; see, *inter alia*, Ehlers and Sprumont (2008) and Section 5.1 below.

As a notable shortcoming, however, the top-cycle rule allows one choose an item w in S even though another item in that menu is strictly preferred to w for every member of $\mathcal{P}$.[3] As such, one cannot consider (1) as a model of "rational choice." Yet, there seems to be a quick fix for this: just drop the dominated alternatives from $\bigcirc(S, \mathbf{R})$, and work with

$$C(S) = \mathbf{MAX}(\bigcirc(S, \mathbf{R}), \succsim), \tag{2}$$

where, for any nonempty set A of alternatives, $\mathbf{MAX}(A, \succsim)$ stands for the set of all maximal elements in A relative to $\succsim$. Unfortunately, this introduces a new problem. Once we eliminate the dominated alternatives from $\bigcirc(S, \mathbf{R})$, which leaves us with the menu $T := \mathbf{MAX}(\bigcirc(S, \mathbf{R}), \succsim)$, some of the alternatives in T may be strictly inferior to *all* of the other alternatives in T with respect to the transitive closure of the restriction of $\mathbf{R}$ to T. That is, not every element of T is maximal with respect to the said relation, betraying the very logic of the top-cycle choice rule.[4] We should instead look at the top-cycle choices in T with respect to $\mathbf{R}$, and this prompts:

$$C(S) = \bigcirc(\mathbf{MAX}(\bigcirc(S, \mathbf{R}), \succsim), \mathbf{R}). \tag{3}$$

Fortunately, this two-way reasoning stops here. Any item in this set is undominated with respect to $\succsim$ and it cannot be eliminated on the basis of $\mathbf{R}$. Moreover, while (3) describes a three-stage choice procedure that does not seem to be easy to work with, it can actually be written as a two-stage model simply by changing the order of application of the $\mathbf{MAX}$ and $\bigcirc$ operators in (2). That is, (3) can equivalently be written as:

$$C(S) = \bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}). \tag{4}$$

This unifies several choice theories based on nontransitive and incomplete preferences. In particular, if $\mathcal{P}$ is a singleton, then $\mathbf{R}$ is equal to $\succsim$ by our consistency assumptions, and $C(S)$ in (4) consists of all $\succsim$-maxim*um* elements in S; this is the standard rational choice model. If $\mathcal{P}$ is such that there are no dominated alternatives in S, then (4) reduces to

[3] Let S consist of the alternatives x, y, z and w, and suppose $\mathcal{P} = \{\succsim_1, \succsim_2, \succsim_3\}$ where $x \succ_1 w \succ_1 y \succ_1 z$, $z \succ_2 x \succ_2 w \succ_2 y$ and $y \succ_3 z \succ_3 x \succ_3 w$. Then, where $\mathbf{R}$ is obtained from $\mathcal{P}$ by the simple majority rule, $\bigcirc(S, \mathbf{R}) = S$, while $x \succ_i w$ for each $i = 1, 2, 3$. (Here, $\succ_i$ stands for the strict part of $\succsim_i$ for each i.)

[4] Let S consist of the alternatives x, y, z, w and w', and suppose $\mathcal{P} = \{\succsim_1, \succsim_2, \succsim_3\}$ where $x \succ_1 w \succ_1 y \succ_1 w' \succ_1 z$, $z \succ_2 w' \succ_2 x \succ_2 w \succ_2 y$ and $y \succ_3 z \succ_3 x \succ_3 w' \succ_3 w$. Then, where $\mathbf{R}$ is obtained from $\mathcal{P}$ by the simple majority rule, we have $\bigcirc(S, \mathbf{R}) = S$, and hence $\mathbf{MAX}(\bigcirc(S, \mathbf{R}), \succsim) = \{x, y, z, w'\}$. And yet w' does not belong to the top-cycle in $\{x, y, z, w'\}$ with respect to $\mathbf{R}$.

the top-cycle choice rule (1). Similarly, if $\mathbf{R}$ renders any two $\succsim$-incomparable alternatives indifferent, then (4) becomes $C(S) = \mathbf{MAX}(S, \succsim)$. Thus, our model also generalizes rational choice theory with incomplete preferences as developed in, say, Eliaz and Ok (2006). In what follows, we refer to this particular case of our model as the *Pareto choice rule*.

The primary purpose of the present paper is to study the behavioral properties of choice correspondences that can be represented as in (4), but with an additional consistency condition that connects $\succsim$ and $\mathbf{R}$ in the following way:

$$x \succsim y \mathbf{R} z \quad \text{implies} \quad x \mathbf{R} z \tag{5}$$

(and similarly, $x \mathbf{R} y \succsim z$ implies $x \mathbf{R} z$). This is best viewed as a condition that limits the extent of $\mathbf{R}$'s nontransitivity; it says simply that if y is deemed better than z (for whatever reason), and x is unambiguously better than y (because every single member of $\mathcal{P}$ says exactly this), then x would be chosen over z. This property is trivially satisfied when $\mathbf{R}$ is transitive, and in all three of the scenarios we considered in our opening paragraph.

Two binary relations $\succsim$ and $\mathbf{R}$ on an alternative space constitute a *weak preference structure* if $\succsim$ is transitive, and $\mathbf{R}$ is a complete superrelation of $\succsim$ such that (5) holds for all alternatives x, y and z. (When $\mathbf{R}$ is also transitive, we refer to the pair $(\succsim, \mathbf{R})$ as a *transitive weak preference structure*.) If, in addition, $\mathbf{R}$ is an extension of $\succsim$, we say that $(\succsim, \mathbf{R})$ is a *preference structure*. These concepts were recently introduced in Nishimura and Ok (2019), where a choice correspondence C is said to be *rationalized by* $(\succsim, \mathbf{R})$ provided that (4) holds for all feasible menus S, and $(\succsim, \mathbf{R})$ is a weak preference structure. Our discussion so far is only meant to be an alternative, heuristic motivation for this choice model.[5] After going through a few preliminaries, we describe this model in Section 4 formally, and motivate it further.

Nishimura and Ok (2019) have shown that a choice correspondence that is rationalized by a (weak) preference structure is nonempty-valued under fairly general conditions, and that many choice models considered in the literature can be represented in this manner. However, the predictive content of this choice model, or a lack thereof, remains unexplored. In particular, testable behavioral implications of the model are yet to be identified, and it is not clear how far off they are from the Weak Axiom of Revealed Preference (WARP). As such, it is not plain if one can regard this model as a tenable "rational choice model."

We attack this problem here by introducing a purely behavioral characterization of choice correspondences rationalized by preference structures (may these be proper, weak or transitive weak). Our results are based on four behavioral principles, each corresponding to a relaxation of the α or β-Axioms (and hence of WARP) of classical choice theory (Section 5). We find that, despite its initial appearance, the model (4) with $\succsim$ and $\mathbf{R}$ making up some form of a preference structure, has fairly simple behavioral foundations that closely parallel that of classical rational choice theory. In particular, a relatively well-known, but somewhat understudied, property of choice theory, the *Aizerman Axiom* plays a central role in our characterizations. This axiom says that removing some unchoosable alternatives from a given menu should not influence the agent's behavior, i.e., the set of choosable alternatives must remain the same.

[5] In particular, we have motivated $\succsim$ above as the intersection of a nonempty collection of complete and transitive binary relations, but this is without loss of generality: Any transitive binary relation can be written in this manner (assuming the Axiom of Choice).

Our characterization of the general model of rationalizability provides a minimal set of behavioral properties that one can use to test the validity of this model by choice data collected in, say, experimental environments (Section 5.3.1). Strengthening the associated axioms suitably yields new, and surprisingly simple, characterizations of some special cases such as the top-cycle and Pareto choice rules (Section 5.1 and 5.3.2), while rationalizability by a transitive weak preference structure provides a sharp refinement of the general theory, yielding a choice model with superior predictive power (Section 5.3.2). Furthermore, our analysis uncovers unexpected connections between some of these models. Specifically, we find that rationalizability by a transitive weak preference structure is equivalent to the Pareto choice rule at a behavioral level. This observation provides an alternative perspective on choices induced by transitive weak preference structures, which are (implicitly) adopted in several recent papers on decision theory in the contexts of decision making under risk and uncertainty, as well as time preferences.

In Section 6, we turn to the problem of determining to what extent we can recover the preference structure of an agent from her choice correspondence C (provided, of course, C is rationalizable by at least one preference structure). In the standard theory this issue hardly arises, for a choice correspondence can be rationalized by at most one complete and transitive preference relation (provided that the choice domain is sufficiently rich). The present situation is more complicated. In general, a choice correspondence can be rationalized by multiple preference structures with different core preferences (but the revealed parts of these structures must be identical). Thus, if rationalizable, C may well "reveal" a *family* of core preferences, each member of which can be coupled with a fixed (unique) revealed preference $\mathbf{R}$ to form a preference structure that rationalizes C. Nishimura and Ok (2019) prove that ordering this set on the basis of decisiveness of core preferences yields a complete lattice, and identify the most decisive of all these preferences explicitly. In turn, we provide here a complete characterization of all preference structures (with, and without, the transitivity requirement on $\mathbf{R}$) that may rationalize a given choice correspondence (Sections 6.1 and 6.2). This provides an explicit view of the extent to which one's (unobservable) core preferences can be determined from her (observable) choice behavior.

The basic setting of the present paper is, by necessity, that of abstract choice theory. To illustrate the nature of our general findings, and to hint at their use in concrete environments, we use a particularly simple model of decision making under uncertainty (Section 4.3). This model is used at various points of our exposition to highlight some fine points of the notion of "rationalization by a preference structure," and to connect this notion with earlier work on decision theory in which this notion was used implicitly (albeit, with transitive revealed preferences). It also serves to show that, in certain economic environments, one's choice behavior may "reveal" her core preferences in a much sharper way than our general uniqueness theorems indicate (Section 6.3).

2 Nomenclature

Throughout the exposition, X stands for an arbitrary nonempty set, unless otherwise is explicitly stated. This section collects the basic terminology we use in the paper that pertains to the general theory of binary relations on X.

Binary Relations. A **binary relation** $\mathbf{R}$ on X is a nonempty subset of $X \times X$, but one writes $x \mathbf{R} y$ instead of $(x,y) \in \mathbf{R}$. For any nonempty $A \subseteq X$, by $A \mathbf{R} y$ we mean $x \mathbf{R} y$ for every $x \in A$; the expression $y \mathbf{R} A$ is similarly understood. The **restriction** of $\mathbf{R}$ to A is defined as $\mathbf{R}|_A := \mathbf{R} \cap (A \times A)$.

The **asymmetric** (or **strict**) **part** of a binary relation $\mathbf{R}$ on X is defined as the binary relation $\mathbf{R}^>$ on X with $x \mathbf{R}^> y$ iff $x \mathbf{R} y$ and not $y \mathbf{R} x$, while the **symmetric part** of $\mathbf{R}$ is given by $\mathbf{R}^= := \mathbf{R} \backslash \mathbf{R}^>$. When either $x \mathbf{R} y$ or $y \mathbf{R} x$, we say that x and y are **R-comparable**. The collection of all pairs (x,y) that are not $\mathbf{R}$-comparable is denoted by $\mathrm{Inc}(\mathbf{R})$, that is, $(x,y) \in \mathrm{Inc}(\mathbf{R})$ iff neither $x \mathbf{R} y$ nor $y \mathbf{R} x$. If $\mathrm{Inc}(\mathbf{R}) = \varnothing$, we say that $\mathbf{R}$ is **complete** (or **total**).

Given a pair of binary relations $\mathbf{R}$ and $\mathbf{Q}$ on X, we simply write $x \mathbf{R} y \mathbf{Q} z$ to mean $x \mathbf{R} y$ and $y \mathbf{Q} z$. The **composition** of $\mathbf{R}$ and $\mathbf{Q}$ is also a binary relation defined by $\mathbf{R} \circ \mathbf{Q} := \{(x,y) \in X \times X : x \mathbf{R} z \mathbf{Q} y \text{ for some } z \in X\}$. We say that $\mathbf{Q}$ is a **subrelation** of $\mathbf{R}$, and that $\mathbf{R}$ is a **superrelation** of $\mathbf{Q}$, if $\mathbf{Q} \subseteq \mathbf{R}$.

We denote the diagonal of $X \times X$ by $\triangle_X$, that is, $\triangle_X := \{(x,x) : x \in X\}$. A binary relation $\mathbf{R}$ on X is said to be **reflexive** if $\triangle_X \subseteq \mathbf{R}$, **antisymmetric** if $\mathbf{R}^= \subseteq \triangle_X$, and **transitive** if $\mathbf{R} \circ \mathbf{R} \subseteq \mathbf{R}$. A **preorder** (or a *preference relation*) on X refers to a reflexive and transitive binary relation. Throughout the paper, $\succsim$ and $\trianglerighteq$ denote generic preorders with asymmetric parts $\succ$ and $\triangleright$, respectively.

The **transitive closure** of a binary relation $\mathbf{R}$ on X is the smallest transitive superrelation of $\mathbf{R}$, denoted as $\mathrm{tran}(\mathbf{R})$. This relation always exists: we have $x \,\mathrm{tran}(\mathbf{R})\, y$ iff there exist a $k \in \mathbb{Z}_+$ and $x_0, ..., x_k \in X$ such that $x = x_0 \mathbf{R} x_1 \mathbf{R} \cdots \mathbf{R} x_k = y$. Obviously, $\mathrm{tran}(\mathbf{R})$ is a preorder on X, provided that $\mathbf{R}$ is reflexive. Moreover, $\mathrm{tran}(\mathbf{R})^> \subseteq \mathbf{R}^>$ whenever $\mathbf{R}$ is complete, but the converse inclusion does not hold in general.

Extension of Binary Relations. Let $\mathbf{R}$ be a binary relation on X. If $\mathbf{Q}$ and $\mathbf{Q}^>$ are subrelations of $\mathbf{R}$ and $\mathbf{R}^>$, respectively, we say that $\mathbf{R}$ is an **extension** of $\mathbf{Q}$ (or that $\mathbf{R}$ *extends* $\mathbf{Q}$). If $\mathbf{R}$ is total and extends $\mathbf{Q}$, we refer to it as a **completion** of $\mathbf{Q}$.

Transitivity with Respect to Another Binary Relation. A useful concept in the analysis of nontransitive binary relations is the notion of *transitivity with respect to a binary relation.* Put precisely, given any two binary relations $\mathbf{R}$ and $\mathbf{Q}$ on X, we say that $\mathbf{R}$ is **Q-transitive** if $\mathbf{R} \circ \mathbf{Q} \subseteq \mathbf{R}$ and $\mathbf{Q} \circ \mathbf{R} \subseteq \mathbf{R}$, which means that either $x \mathbf{R} y \mathbf{Q} z$ or $x \mathbf{Q} y \mathbf{R} z$ implies $x \mathbf{R} z$ for any $x, y, z \in X$. This notion generalizes the classical concept of transitivity, for, obviously, $\mathbf{R}$ is $\mathbf{R}$-transitive iff it is transitive.

Extrema of Binary Relations. Let $\mathbf{R}$ be a binary relation on X, and take any nonempty subset S of X. We say that an element x of S is **R-maximal** in S if there is no $y \in S$ with $y \mathbf{R}^> x$, and **R-maximum** in S if $x \mathbf{R} S$ (that is, $x \mathbf{R} y$ for every $y \in S$). We denote the set of all $\mathbf{R}$-maximal and $\mathbf{R}$-maximum elements in S by $\mathbf{MAX}(S, \mathbf{R})$ and $\max(S, \mathbf{R})$, respectively. It is plain that $\max(S, \mathbf{R}) \subseteq \mathbf{MAX}(S, \mathbf{R})$, but this inequality may hold strictly (unless $\mathbf{R}$ is complete).

3 Choice and Top-Cycles

In what follows, we view X as the collection of all mutually exclusive choice prospects for an economic agent (who may itself be a collection of individuals, such as a board of directors, congress, or a family). In turn, $\mathfrak{X}$ stands for the collection of all nonempty *finite* subsets of X; as usual, each member of $\mathfrak{X}$ is thought of as a feasible menu of choice alternatives. We focus on finite menus to simplify the analysis. Following a topological approach, the online appendix of this paper extends our main findings in a way that allows for infinite menus.

Choice Correspondences. A **choice correspondence** on $\mathfrak{X}$ is a map $C : \mathfrak{X} \to 2^X$ such that $\varnothing \neq C(S) \subseteq S$ for every $S \in \mathfrak{X}$. As usual, for any $S \in \mathfrak{X}$, we wish to interpret $C(S)$ as the set of all alternatives the agent deems "choosable" in the feasible menu S.

Top-Cycles. When a given binary relation $\mathbf{R}$ on X is nontransitive, $\mathbf{MAX}(S, \mathbf{R})$ may be empty even if S is finite. For this reason, alternative notions of extrema are developed for such binary relations. Among these, the best known is the notion of *top-cycle* to which we now turn, in the context of complete binary relations.

We say that a nonempty subset A of S is an **$\mathbf{R}$-highset in S** if

$$x \ \mathbf{R}^{>} \ y \quad \text{for every } x \in A \text{ and } y \in S \backslash A.$$

Notice that the collection of all $\mathbf{R}$-highsets in S is nonempty, because it contains S. Besides, it is readily checked that this collection is linearly ordered by set inclusion $\supseteq$. Consequently, if it exists, there is a unique smallest (with respect to $\supseteq$) $\mathbf{R}$-highset in S, and this set equals $\bigcap\{A \in 2^X : A \text{ is an } \mathbf{R}\text{-highset in } S\}$. We thus define the **top-cycle in S with respect to $\mathbf{R}$**, which we denote by $\bigcirc(S, \mathbf{R})$, as

$$\bigcirc(S, \mathbf{R}) := \bigcap\{A \in 2^X : A \text{ is an } \mathbf{R}\text{-highset in } S\}.$$

This set is nonempty iff the smallest $\mathbf{R}$-highset in S exists.

A nonempty subset A of X is an **$\mathbf{R}$-cycle** if for every $x, y \in A$, we can find finitely many elements $a_1, ..., a_k$ of A such that $x \ \mathbf{R} \ a_1 \ \mathbf{R} \cdots \mathbf{R} \ a_k \ \mathbf{R} \ y$. It is not difficult to show that, whenever it is nonempty, $\bigcirc(S, \mathbf{R})$ is an $\mathbf{R}$-cycle which is also an $\mathbf{R}$-highset in S. In fact, $\bigcirc(S, \mathbf{R})$ is the only such $\mathbf{R}$-cycle in S, so we can characterize it as such. Moreover, one can show that $\bigcirc(S, \mathbf{R})$ obtains upon the maximization of the transitive closure of $\mathbf{R}|_S$ on S. As these characterizations are well-known, we omit their proofs here, but state them formally for future reference.

Lemma 3.1. *Let S be a nonempty subset of X, and $\mathbf{R}$ a complete binary relation on X. Then, either $\bigcirc(S, \mathbf{R}) = \varnothing$ or $\bigcirc(S, \mathbf{R})$ is the unique $\mathbf{R}$-cycle which is also an $\mathbf{R}$-highset in S. Moreover, we have*

$$\bigcirc(S, \mathbf{R}) = \max(S, \operatorname{tran}(\mathbf{R}|_S)).$$

Note that $\bigcirc(S, \mathbf{R}) \neq \varnothing$ for a finite set S. More generally, one can show that $\bigcirc(S, \mathbf{R})$ is nonempty when X is a topological space, S is compact, and $\mathbf{R}$ is upper semicontinuous (in the sense that $\{y \in X : y \ \mathbf{R} \ x\}$ is closed in X for every $x \in X$).[6]

[6]See, for instance, Duggan (2007).

Top-Cycle Choice Rules. A choice correspondence C on $\mathfrak{X}$ is said to be a **top-cycle choice rule** if there exists a complete binary relation $\mathbf{R}$ on X such that

$$C(S) = \bigcirc(S, \mathbf{R}) \quad \text{for every } S \in \mathfrak{X}.$$

Such rules are commonly used in the theory of social choice and tournaments (cf. Kalai and Schmeidler (1977), Schwartz (1986), Laslier (1997), and Ehlers and Sprumont (2008).)

4 Choice and Preference Structures

4.1 Preference Structures

A **weak preference structure** on X is an ordered pair $(\succsim, \mathbf{R})$ where $\succsim$ is a preorder on X and $\mathbf{R}$ is a $\succsim$-transitive and complete binary relation on X with $\succsim \subseteq \mathbf{R}$. In this context, we refer to $\succsim$ as the **core preference relation** of the structure, and to $\mathbf{R}$ as its **revealed preference relation**. If $\mathbf{R}$ is transitive, we refer to $(\succsim, \mathbf{R})$ as a **transitive weak preference structure** on X, and if, instead, $\mathbf{R}$ is an extension of $\succsim$ (so that $\succ \subseteq \mathbf{R}^{>}$), we refer to it simply as a **preference structure**. (If we need to distinguish a preference structure as not being weak, we refer to it below as a *proper* preference structure.)

The interpretation of $(\succsim, \mathbf{R})$ is as follows. The agent in question is entirely confident in the preferential ranking of *some* of the alternatives in X, and these are modeled by $\succsim$. Indeed, the solution of some pairwise choice problems may just appear "obvious" to her. (For instance, a vegetarian is likely to make her choice from the menu {steak, a vegetarian dish} without any qualms, while her choice from the menu of two distinct vegetarian dishes may not be obvious, and may depend on her mood.) As we have suggested in the Introduction, $\succsim$ may also arise because the agent bases her decisions on the advices that she is given by a set of consultants who all agree with the rankings stipulated by $\succsim$, or because she has a set of potentially conflicting selves/moods about the alternatives, but when $\succsim$ applies, all of her selves/mood agree with it, etc. Or, the agent may actually be a social decision maker on behalf of a collection of individuals (each with her own preference relation), and $\succsim$ may correspond to the ranking of the alternatives according to the Pareto rule. Regardless of where it comes from, it seems reasonable to suppose $\succsim$ be transitive as the agent is unlikely to make cyclical choices across pairwise choice problems that she finds "straightforward" to resolve. However, as she may well be conflicted about the comparison of some alternatives, we do not posit $\succsim$ be complete, thereby deviating from the standard theory of rational choice. (When $\succsim$ is complete, $\mathbf{R}$ must equal $\succsim$, and hence the model reduces to the standard theory of complete preference relations.)

When confronted with the problem of choosing between any two alternatives x and y, the agent will have to render a decision, whether or not she finds this decision "straightforward." These decisions yield the (observable) revealed preferences of the agent, which are modeled through $\mathbf{R}$. Its interpretation mandates that $\mathbf{R}$ be complete, but it is presumed that "hard choices" may result in cyclical choice patterns, so $\mathbf{R}$ is permitted to be nontransitive. For example, if our economic agent is a social planner who acts on behalf of a set of individuals, then the standard methods of aggregating constituent preferences (such as majority voting) may result in the revelation of nontransitive rankings of the alternatives.

For our interpretation of the model $(\succsim, \mathbf{R})$ to hold water, $\succsim$ and $\mathbf{R}$ must be consistent. This is ensured through two channels. First, it is assumed that $\mathbf{R}$ is a superrelation of $\succsim$. Thus, $x \succsim y$ (x is "surely" better than y) implies $x \mathbf{R} y$ (we observe the agent choose x over y). If $(\succsim, \mathbf{R})$ is proper, we additionally require that $x \succ y$ (x is "surely" strictly better than y) imply $x \mathbf{R}^{>} y$ (we observe the agent never choose y over x). Second, it is assumed that $\mathbf{R}$ is transitive with respect to $\succsim$. (To wit, suppose $x \mathbf{R} y$ and $y \succsim z$ for some x, y and z in X. We interpret this as saying that the agent likes x better than y, even though she may be somewhat insecure about this decision, while she prefers y over z in complete confidence. It then stands to reason that the "obvious" superiority of y over z would entail that she would like x better than z (that is $x \mathbf{R} z$), but of course, she may be "uneasy" about this conclusion (that is, $x \succsim z$ need not hold).)

The model of preference structures is introduced recently by Nishimura and Ok (2019) who have shown that this model captures phenomena like rational choice, indecisiveness, imperfect ability of discrimination, regret, and advise taking, among others. We refer the reader to that paper for a detailed analysis of preference structures, and turn our attention here to how one's choices may be rationalized by means of such structures.[7]

4.2 Rationalization by Preference Structures

For any weak preference structure $(\succsim, \mathbf{R})$ on X, we say that a choice correspondence C on $\mathfrak{X}$ is **rationalized by** $(\succsim, \mathbf{R})$ if

$$C(S) = \bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}) \quad \text{for every } S \in \mathfrak{X}. \tag{6}$$

This model describes an economic agent who makes her choice(s) from a given feasible menu S by employing a two-step procedure. First, she looks for the alternatives in S that are maximal relative to her core preference $\succsim$. If there is only one such alternative in S, then she chooses that alternative. If there is a multiplicity of such alternatives (which may be due to indifferences and/or incomparabilities instigated by $\succsim$), she evaluates those alternatives on the basis of her second (complete) binary relation $\mathbf{R}$. She finalizes her choice(s) by maximizing $\mathbf{R}$ on $\mathbf{MAX}(S, \succsim)$ in the sense of finding the top-cycle in $\mathbf{MAX}(S, \succsim)$ with respect to $\mathbf{R}$; this is the set of all alternatives she deems "choosable" in S.

Rationalizability by a preference structure $(\succsim, \mathbf{R})$ generalizes a number of rationalizability notions encountered in choice theory. In particular, if $\succsim$ equals $\mathbf{R}$, (6) becomes

$$C(S) = \max(S, \succsim) \quad \text{for every } S \in \mathfrak{X}, \tag{7}$$

which is but the classical notion of rationalizability. Second, if $\mathbf{R}$ satisfies $x \mathbf{R} y$ iff either $x \succsim y$ or x and y are not $\succsim$-comparable, (6) becomes

$$C(S) = \mathbf{MAX}(S, \succsim) \quad \text{for every } S \in \mathfrak{X}.$$

[7] Giarlotta and Watson (2018) also report some basic mathematical results about preference structures, but their work is mostly motivated by operations research type goals (that are not directly related to choice theory). In that paper a weak preference structure is referred to as a *complete bi-preference*, and a preference structure as a *monotonic complete bi-preference.*

We thus see that the choice theory based on preference structures generalizes that based on incomplete (but transitive) preference relations (as developed in, say, Eliaz and Ok (2006)). Third, if $\succsim$ equals $\triangle_X$ (which means that the agent cannot compare any two distinct alternatives on the basis of her core preference), then (6) says that C is the top-cycle choice rule induced by $\mathbf{R}$. Thus, obviously, the choice theory based on preference structures subsumes top-cycle choice rules.

Finally, and mainly to give a nontrivial example, let $u : X \to \mathbb{R}$ be any function and take any real number $\varepsilon \geq 0$. Define the binary relation $\mathbf{R}$ on X as $x\ \mathbf{R}\ y$ iff $u(x) \geq u(y) - \varepsilon$. Note that $\mathbf{R}^>$ is then a canonical semiorder with $x\ \mathbf{R}^>\ y$ iff $u(x) > u(y) + \varepsilon$ and $\mathbf{R}^=$ a canonical similarity relation with $x\ \mathbf{R}^=\ y$ iff $|u(x) - u(y)| \leq \varepsilon$. (The idea is that the agent does not discriminate between alternatives whose utility values are close enough; Luce (1956) thus refers to the number ε as *just noticeable difference*.) Next, consider the preorder $\succsim$ on X defined by $x \succsim y$ iff either $x = y$ or $u(x) > u(y) + \varepsilon$. (The interpretation is that the pairwise ranking of any two alternatives is an "easy" one if the utilities of these alternatives are sufficiently distinct, and "hard" otherwise.) It is plain that $(\succsim, \mathbf{R})$ is a preference structure on X. It is also quite easy to prove that (6) reads as

$$C(S) = \{x \in S : \sup u(S) - u(x) \leq \varepsilon\} \quad \text{for every } S \in \mathfrak{X}$$

for this preference structure. Thus, the choice theory based on preference structures generalizes that of *constant threshold choice models* (studied by, say, Luce (1956), Jamison and Lau (1973, 1975), Fishburn (1975), and Beja and Gilboa (1992)).

For other examples of choice correspondences that are rationalizable by preference structures, we refer the reader to Nishimura and Ok (2019).

The Case of Single-Valued Choice Correspondences. A choice correspondence on $\mathfrak{X}$ is said to be **single-valued** if $|C(S)| = 1$ for every $S \in \mathfrak{X}$. One can show that if such a choice correspondence is rationalizable by a weak preference structure $(\succsim, \mathbf{R})$, then it must be rationalizable in the standard sense. Put differently, rationalization by a *preference relation* and that by a *weak preference structure* are equivalent concepts for single-valued choice correspondences.

Top-Cycles of Maxima vs. Maxima of Top-Cycles. In our Introduction, we have noted that $S \mapsto \mathbf{MAX}(\bigcirc(S, \mathbf{R}), \succsim)$ is an interesting choice model, but that an element of $\mathbf{MAX}(\bigcirc(S, \mathbf{R}), \succsim)$ may not be maximal with respect to the transitive closure of the restriction of $\mathbf{R}$ to this set. In other words, there may be a strictly smaller $\mathbf{R}$-highset in $\mathbf{MAX}(\bigcirc(S, \mathbf{R}), \succsim)$, betraying the very logic of the top-cycle choice rule. As such, a more natural choice model that is alternative to the one we introduced above is $S \mapsto \bigcirc(\mathbf{MAX}(\bigcirc(S, \mathbf{R}), \succsim))$. And yet, as we have claimed in the Introduction, this model is identical to the rationalization notion (6), at least in the context of finite menus. We now state this fact formally, and prove it in the Appendix.

Proposition 4.1. *Let $(\succsim, \mathbf{R})$ be a weak preference structure on X. Then,*

$$\bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}) = \bigcirc(\mathbf{MAX}(\bigcirc(S, \mathbf{R}), \succsim), \mathbf{R}) \quad \textit{for every } S \in \mathcal{X}.$$

"Rationality" of Rationalization by Preference Structures. If the choice correspondence of the agent is rationalized by a weak preference structure $(\succsim, \mathbf{R})$, what can we say about the alternatives that she deems not worthy of choice? In what sense such an alternative is suboptimal for the agent? The following observation provides an answer to this query, thereby providing a normative motivation for the concept of rationalization by a preference structure.

Proposition 4.2. *Let C be a choice correspondence on $\mathfrak{X}$ rationalized by a weak preference structure $(\succsim, \mathbf{R})$ on X. Then, for any nonempty finite subset S of X and any $y \in S \backslash C(S)$, either $x \succ y$ for some $x \in C(S)$ or $C(S)\ \mathbf{R}^{>}\ y$.*

Suppose we observe that the agent deems an alternative y unchoosable in a feasible (finite) menu S. Proposition 4.2 says that this means one of two things. First, it may be the case that there is a feasible, in fact, choosable, alternative in S that dominates y according to the core preferences of the agent. (If, for instance, the core preference arises from the preferences of a collection of experts, this means that *all* of the experts say that x is better than y, and at least one says that x is strictly better than y.) Second, if this is not the case, that is, none of the choosable alternatives dominate y, then we have $C(S)\ \mathbf{R}^{>}\ y$. Hence, any one of the choosable alternatives in S is *revealed* superior to y in the sense that when pitted against any such alternative (in the context of a pairwise choice problem), we will never observe y be chosen. As such, choices of the agent in S are duly consistent with her pairwise choices relative to all unchosen, but undominated, alternatives in S. It thus seems like there is quite a bit of "rationality" behind the notion of rationalization by a preference structure, even though a preference structure allows for both indecisiveness and nontransitivity on the part of the decision maker. We will corroborate this point in Section 5 by means of a behavioral characterization of such choice procedures.

Remark. Proposition 4.1 remains true even when $\mathbf{R}$ is not $\succsim$-transitive, but this property is essential for the validity of Proposition 4.2. Moreover, finiteness hypothesis can be substantially relaxed in these results. In particular, they both remain true when C is defined on all subsets of X, provided that $\succsim$ is a *Noetherian* preorder, that is, for every nonempty $S \subseteq X$ and every non-$\succsim$-maximal y in S, there exists a $\succsim$-maximal x in S with $x \succ y$.

4.3 Rationalization by Transitive Preference Structures

Let $(\succsim, \mathbf{R})$ be a weak preference structure on X such that $\mathbf{R}$ is transitive. If this structure is proper (that is, $\mathbf{R}$ is not only a superrelation of $\succsim$, but it is actually an extension of it), then $\bigcirc(\mathbf{MAX}(\cdot, \succsim), \mathbf{R})$ is none other than $\max(\cdot, \mathbf{R})$, so our choice model reduces to the classical rational choice theory. However, there are many weak preference structures that are considered in the literature on decision theory in which the revealed preference relation $\mathbf{R}$, while transitive, is not an extension of $\succsim$. In this sort of a situation, the choice model at hand does not simply maximize $\mathbf{R}$; it is indeed distinct from the classical choice theory.

When $(\succsim, \mathbf{R})$ is a transitive weak preference structure on X, (6) becomes

$$C(S) = \max(\mathbf{MAX}(S, \succsim), \mathbf{R}) \quad \text{for every } S \in \mathfrak{X}. \tag{8}$$

While transitivity of $\mathbf{R}$ ensures that $\max(\mathbf{MAX}(S, \succsim), \mathbf{R}) \subseteq \max(S, \mathbf{R})$ for any $S \in \mathfrak{X}$, it is important to note that $\max(\mathbf{MAX}(S, \succsim), \mathbf{R})$ maybe much smaller than $\max(S, \mathbf{R})$. As a matter of fact, thanks to the transitivity of $\mathbf{R}$, here we have

$$C(S) = \mathbf{MAX}(S, \succsim) \bigcap \max(S, \mathbf{R}) \quad \text{for every } S \in \mathfrak{X}, \tag{9}$$

a formula which makes the computation of C particularly easy. One should thus really think of the model (8) as a refinement of both the classical choice model rationalized by the complete and transitive preference relation $\mathbf{R}$, and the choice model that looks at the maximal elements with respect to the core dominance relation $\succsim$. To drive this point home, we next provide an economic application that demonstrates it concretely. We will return to this application later at various points in our exposition.

Remark. Let S be any nonempty subset of X. Then, if (8) holds, so does (9), provided that for every non-$\succsim$-maximal element y in S, there is a $\succsim$-maximal $x \in S$ with $x \succ y$. The latter condition holds trivially when S is finite, but it holds in other contexts as well. In addition to the case of a Noetherian preorder mentioned earlier, if X is a topological space and $\succsim$ is a closed preorder on X, the property in question holds for any nonempty compact $S \subseteq X$. Thus, (8) implies (9) in this situation as well.

An Application to Decision-Making Under Uncertainty. In a well-known paper, Gilboa et al. (2010) provide an interpretation of the famous maxmin expected utility model as an aggregation of the expected utility rankings of a collection of experts who have the same (risk) preferences, but differ over the prior likelihoods of the states of nature. This way of looking at things yields a transitive weak preference structure, and very much suggests adopting the choice model (8).[8]

To simplify our illustration, we look at the Gilboa et al. (2010) model with finitely many states and real-valued acts. Thus, where n is a positive integer, we label the states of nature from 1 to n, and view $\mathbb{R}^n$ as the set of all acts. Let Π be a nonempty collection of probability n-vectors, and define the preorder $\succsim_\Pi$ on $\mathbb{R}^n$ by

$$x \succsim_\Pi y \quad \text{iff} \quad \pi x \geq \pi y \text{ for every } \pi \in \Pi.$$

This preorder is often called the *Bewley preferences with the prior set* Π. Now consider the complete preorder $\mathbf{R}_\Pi$ with

$$x \ \mathbf{R}_\Pi \ y \quad \text{iff} \quad \min_{\pi \in \Pi} \pi x \geq \min_{\pi \in \Pi} \pi y.$$

This is, of course, the *Gilboa-Schmeidler maxmin preferences with the prior set* Π. A decision-theoretic characterization of $\succsim_\Pi$ was provided first by Bewley (1986), and that of

[8] Many other papers in the literature portrait the same flavor. In the context of risk, for instance, the cautious expected utility model of Cerreia-Vioglio, et al. (2015) is interpreted as an aggregation of a given collection of expected utility functions, each of which may be thought of as representing the preferences of an expert or a consultant (over lotteries). Taking $\succsim$ as the intersection of these preferences, and $\mathbf{R}$ as represented by the cautious expected utility induced by the given set of utility functions, we obtain a transitive weak preference structure. In fact, Cerreia-Vioglio, et al. (2015) explicitly suggest using (8) to model the choices of the agent. A similar situation arises in Chambers and Echenique (2018) in the context of time preferences (where the expert opinions arise from differing discount factors).

$\mathbf{R}_\Pi$ was given first by Gilboa and Schmeidler (1989). In turn, Gilboa et al. (2010) provide a joint axiomatization of the relations $\succsim_\Pi$ and $\mathbf{R}_\Pi$ (with the same set of priors).

The upshot here is that $(\succsim_\Pi, \mathbf{R}_\Pi)$ is a transitive weak preference structure. And indeed, this pair is interpreted by Gilboa et al. (2010) in precisely the way we interpret preference structures in general. As such, the choice model to adopt in this context appears to be $S \mapsto \max(\mathbf{MAX}(S, \succsim_\Pi), \mathbf{R}_\Pi)$, which is also tacitly suggested by Gilboa et al. (2010). However, while the applications of the maxmin expected utility model abound, we are not aware of any work that actually adopts this model. It seems that most authors view the work of Gilboa et al. (2010) as merely an "interpretation" of maxmin preferences as a cautious/pessimistic aggregation of Bewley preferences, and continue to use $S \mapsto \max(S, \mathbf{R}_\Pi)$ as the choice model. In fact, not only is the structure of the former model is more in concert with the main interpretation advanced in Gilboa et al. (2010), but it also provides sharper predictions than maximizing either the Bewley or the Gilboa-Schmeidler preferences. As a matter of fact, as (9) witnesses, that model is based on the maximization of *both* of these preferences. The following example is meant to demonstrate this concretely.

Example. Assume that there are three states ($n = 3$), and put $\Pi := \{\pi^1, \pi^2, \pi^3\}$, where $\pi^1 := (\frac{2}{3}, 0, \frac{1}{3})$, $\pi^2 := (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $\pi^3 := (0, 0, 1)$. We consider the feasible menu $S := \{x \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 \leq 1\}$, and put $B := \{x \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 = 1\}$. Then, one can show that maximizing the associated Bewley and Gilboa-Schmeidler preferences yields

$$\mathbf{MAX}(S, \succsim_\Pi) = \{x \in B : x_2 = 0\} \text{ and } \max(S, \mathbf{R}_\Pi) = \left\{x \in B : x_3 \geq \max\left\{\tfrac{1}{3}, 1 - 2x_1\right\}\right\}.$$

Consequently, by (9),

$$\max(\mathbf{MAX}(S, \succsim_\Pi), \mathbf{R}_\Pi) = \left\{x \in B : x_1 \leq \tfrac{2}{3} \text{ and } x_2 = 0\right\}.$$

(See also the Remark above.) Thus, as Figure 1 depicts, rationalization with the transitive weak preference structure $(\succsim_\Pi, \mathbf{R}_\Pi)$ yields a much more refined choice set than maximizing either $\succsim_\Pi$ or $\mathbf{R}_\Pi$. $\square$

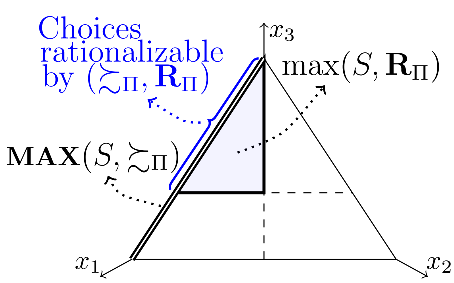

Figure 1: Multi-Prior Example

The fact that the choice model $S \mapsto \max(\mathbf{MAX}(S, \succsim), \mathbf{R})$ refines both $S \mapsto \mathbf{MAX}(S, \succsim)$ and $S \mapsto \max(S, \mathbf{R})$ in the context of transitive weak preference structures is likely to be important in applications. We thus pay special attention to this case below, and provide a behavioral characterization of such choice procedures separately.

5 Revealed Preference Structures

The main objective of this paper is to determine a minimal set of behavioral postulates which characterize choice correspondences that are rationalizable by preference structures. We turn to this issue in this section. But, to streamline the exposition, we first concentrate on the special case of the top-cycle choice rule. Given its importance for social choice theory, our characterization of this choice rule may be of independent interest. At any rate, we use this characterization to motivate the structure of our main characterization of choice correspondences that are rationalizable by preference structures.

5.1 A Characterization of Top-Cycle Choice Rules

The only axiomatic characterizations of top-cycle choice rules we are aware of are those of Ehlers and Sprumont (2008) and Houy (2011). Our characterization of this rule will be based on only two behavioral postulates that are posited on an arbitrarily given choice correspondence C on $\mathfrak{X}$. This characterization is intended to serve as a segue to our main characterization theorems.

To begin, suppose we can partition a feasible menu into two parts such that, in pairwise comparisons, every alternative in the first part is chosen uniquely over every alternative in the second part. Our first axiom says in this case that all choosable items in the grand menu must come from the first part. We formalize this as:

(A1) Preference Consistency. For every $S, T \in \mathfrak{X}$ such that $C\{x, y\} = \{x\}$ for every $(x, y) \in S \times T$, we have $C(S \cup T) \subseteq S$.

This property seems fairly compelling from both the normative and positive viewpoints. We are not aware of a counterpart of it in the context of choice theory, but note that it can be thought of as a (very) special case of *Sen's α-Axiom* (reviewed below).

To introduce our next axiom, let us recall that the classical *Arrow's Choice Axiom* says that, for every $S, T \in \mathfrak{X}$ with $S \subseteq T$ and $C(T) \cap S \neq \varnothing$, we have $C(T) \cap S = C(S)$. It is well-known that this property is equivalent to the classical *Weak Axiom of Revealed Preference* (WARP), and thus characterizes rationalizability of a nonempty-valued choice correspondence by means of a complete preorder. The second axiom that we use here is a straightforward weakening of this property.

(A2) Weak Arrow's Choice Axiom. For every $S, T \in \mathfrak{X}$ with $S \subseteq T$ and $C(T) \cap S \neq \varnothing$, we have $C(T) \cap S \supseteq C(S)$.[9]

The following is a markedly simple characterization of top-cycle choice rules.[10]

Theorem 5.1. *A choice correspondence C on $\mathfrak{X}$ satisfies* (A1) *and* (A2) *if, and only if, it is a top-cycle choice rule.*

[9] While there is no agreed title for it in choice theory, this property is by no means novel. We have learned from Fuad Aleskerov that it was first considered by Aizerman, Zavalishin and Pyatnitski (1977).

[10] By comparison, Ehlers and Sprumont (2008) characterize the top-cycle rule (in the more special case where X is finite) by means of three axioms one of which is weaker than (A1).

Remark. For expositional purposes, we have not stated Theorem 5.1 in its strongest form. In a more general statement, $\mathfrak{X}$ can be taken as the set of all nonempty subsets of X, and $C : \mathfrak{X} \to 2^X$ any map that is nonempty-valued on finite sets. Then, a straightforward modification of the proof we give in the Appendix shows that C satisfies (A1) and (A2) iff there is a complete binary relation $\mathbf{R}$ on X such that $C(S) = \bigcirc(S, \mathbf{R})$ for every $S \in \mathfrak{X}$ with $C(S) \neq \varnothing$.

5.2 Behavioral Choice Axioms for Rationalizability

While this is not entirely trivial, any choice correspondence on $\mathfrak{X}$ that is rationalizable by a preference structure satisfies (A1). Given Theorem 5.1, and because any top-cycle choice rule is rationalizable in this manner, a characterization of these sorts of choice correspondences require us work with weaker properties than (A2). In what follows, we introduce three such properties.

C-**Domination and Irreducibility.** It is easy to see that a choice correspondence C that is rationalized by a preference structure $(\succsim, \mathbf{R})$ on X need not satisfy (A2). Indeed, even the correspondence on $\mathfrak{X}$ that maps any S to $\mathbf{MAX}(S, \succsim)$ may not satisfy this property, unless the preorder $\succsim$ on X is complete. (For example, if x and y, as well as y and z, are not $\succsim$-comparable, but $z \succ x$, we would have $\{x, y\} = C\{x, y\}$ but $\{y, z\} = C\{x, y, z\}$.) On the other hand, if C is rationalized by $(\succsim, \mathbf{R})$, and if S and T are feasible menus with $S \subseteq T$, then an immediate appeal to the second part of Lemma 3.1 shows that we have $C(T) \cap S \supseteq C(S)$, provided that no alternative in T is dominated by any other alternative in S relative to $\succsim$. This suggests that we may still make use of (A2) to characterize rationalizability by a preference structure, provided that we apply this property on menus whose contents do not "dominate" each other in a choice-theoretic sense.

To clarify matters, it will be useful to introduce the following auxiliary notion.

Definition. Let C be a choice correspondence on $\mathfrak{X}$. For any alternatives x and y in X, we say that y C**-dominates** x if x does not belong to $C(T)$ for any $T \in \mathfrak{X}$ with $y \in T$. In turn, a subset S of X is said to be C**-irreducible** if no element of S is C-dominated by an element of S.

In words, if y C-dominates x, we understand that x is never chosen in a feasible menu that contains y. Intuitively, we interpret this situation as saying that y is unambiguously better than x (for the individual with the choice correspondence C). In turn, S is C-irreducible iff no such dominance relation is present between any two members of S, that is, for every x and y in S, there is a menu $A \in \mathfrak{X}$ with $y \in A$ and $x \in C(A)$. For example, in the example of the previous paragraph, $\{x, y\}$ is C-irreducible but $\{x, y, z\}$ is not.

The following postulate weakens (A2) by applying it only to C-irreducible menus.

(B2) Arrow's Undominated Choice Axiom. For every C-irreducible $S, T \in \mathfrak{X}$ with $S \subseteq T$ and $C(T) \cap S \neq \varnothing$, we have $C(T) \cap S \supseteq C(S)$.

We emphasize that while this property has an obvious normative appeal, and is weaker than (A2), it still has a considerable bite. Indeed, it may well be violated in choice situations in which a dominated alternative acts as a reference point. In particular, (B2) rules out behavioral phenomena like the attraction effect and limited attention. To capture this sort of phenomena, one needs to reformulate a menu-dependent choice model by using preference structures. We will not pursue this matter in this paper, however.

The Chernoff Axiom. Perhaps the most well-known postulate to impose on a choice correspondence C on $\mathfrak{X}$ is the *Chernoff Axiom*: For every $S, T \in \mathfrak{X}$ with $S \subseteq T$, we have $C(T) \cap S \subseteq C(S)$.[11] It is easy to verify that the property of rationalizability by a preference structure $(\succsim, \mathbf{R})$ is not strong enough to ensure the satisfaction of this axiom. Intuitively speaking, this is because of the $\mathbf{R}^{>}$-cycles that may be present within the top-cycle in $\mathbf{MAX}(T, \succsim)$ with respect to $\mathbf{R}$. However, if the agent finds at most two alternatives choosable in the larger menu T, then, trivially, there is no such cycle, so any one of these alternatives, if available, must be deemed choosable in the smaller menu S. That is, as we shall show below, rationalizability by a preference structure entails the following special case of the Chernoff Axiom.

(B3) The Binary Chernoff Axiom. For every $S, T \in \mathfrak{X}$ with $S \subseteq T$ and $|C(T)| \leq 2$, we have $C(T) \cap S \subseteq C(S)$.

We can think of this property as saying that if one's choice behavior from a given menu T is fairly resolute (in the sense that potential indecisiveness and/or indifference of this agent does not result in deeming more than two alternatives in T as most desirable), then she would act consistently (in the sense of Chernoff) on all subsets of T.

The Aizerman Condition. Let S and T be two feasible menus with $S \subseteq T$. Recall that Arrow's Choice Axiom says that if $C(T)$ intersects S, then $C(S) = C(T) \cap S$, that is, the choosable alternatives from S are precisely those alternatives in S that were already deemed choosable from the larger menu T. Clearly, this property gets only more appealing if *all* alternatives that are found to be choosable from T remain feasible in the smaller menu S. With this additional prerequisite, Arrow's Choice Axiom becomes:

(B4) The Aizerman Condition.[12] For every $S, T \in \mathfrak{X}$ with $S \subseteq T$ and $C(T) \subseteq S$, we have $C(S) = C(T)$.

In the words of Aleskerov, Bouyssou and Monjardet (2007, p. 40), the Aizerman Condition "... is satisfied by those choice functions where the contraction of T by casting out some or even all alternatives which are not chosen from the initial set T does not change choice." We refer the reader to that book for a detailed discussion of this property in the context of choice theory.

[11] This property was introduced to choice theory first by Chernoff (1954), but it became popular only after the contribution of Sen (1971). Most authors thus refer to it as *Sen's α-Axiom.*

[12] This axiom too appears in Chernoff (1954), but until the works of Mark Aizerman and his group in the early 1980s, it did not play a major role in the theory of choice. It is mainly for this reason that Moulin (1985) has dubbed this property as the *Aizerman Axiom.* (To be precise, Moulin requires only $C(S) \subseteq C(T)$ in his formulation, but couples this property with the Chernoff Axiom. These two properties jointly imply the Aizerman Condition as stated above.)

5.3 Behavioral Characterizations of Rationalizability

5.3.1 Rationalizability by Weak and Proper Preference Structures

We have seen in Section 4.2 that our choice model encompasses a variety of alternative theories considered in the literature, such as the classical theory of rational choice, the top-cycle choice rule, and the constant threshold model, among others. Given this generality, one may be worried that as a theory of choice, rationalization by preference structures may not have much predictive power. The following result shows that this is definitely not the case: Rationalization by a preference structure entails all four of the properties (A1) and (B2)-(B4), and in turn, any choice correspondence that satisfies these properties is rationalizable by a preference structure. Moreover, rationalizability by a weak preference structure and that by a proper preference structure turn out to be equivalent concepts.

Theorem 5.2. *For any choice correspondence C on $\mathfrak{X}$, the following are equivalent*:
(a) *C satisfies Axioms* (A1) *and* (B2)-(B4);
(b) *C is rationalizable by a preference structure*;
(c) *C is rationalizable by a weak preference structure.*

Earlier, Nishimura and Ok (2019) showed that no choice correspondence that fails the Condorcet Criterion (or the so-called *Sen's Axiom-δ*) can be rationalized by a preference structure. Theorem 5.2 identifies several other behavioral implications of this rationalization notion. If one wishes to test whether or not the choices of an agent is rationalizable by a preference structure, it is apparently enough to check for (A1) and (B2)-(B4). Put differently, these axioms provide one with a minimal set of behavioral properties that can be used to test rationalization by preference structures, say in laboratory experiments. The upshot is that insofar as one finds the contents of Axioms (A1) and (B2)-(B4) satisfactory (from either descriptive or normative viewpoints), we can think of the involved choice behavior "as if" this behavior is represented by a preference structure.

5.3.2 Rationalizability by Transitive Weak Preference Structures

As we have noted in Section 4.3, a number of recent papers suggest methods of aggregating a given set of "standard" preference relations (often viewed as the preferences of a collection of experts) in the contexts of decision making under risk and uncertainty, and time preferences. In most of these papers, the aggregation method yields a complete and transitive preference relation which is a superrelation (but not an extension) of the intersection of the given set of preference relations. This furnishes a transitive weak preference structure, and motivates searching for the behavioral foundations for rationalization by such a structure (as in (8)). In other words, the question is how to strengthen (a) of Theorem 5.2 if we demand C be rationalizable by a *transitive* weak preference structure. The answer turns out to be quite clean: Keeping the behavioral properties used in that theorem as is, but strengthening (B3) to the classical Chernoff Axiom, yields the required characterization. (As the Chernoff Axiom already entails (A1), this characterization is actually based on only three properties: (B2), (B4) and the Chernoff Axiom.)

Furthermore, there is an additional twist in the tale. It turns out that for every transitive weak preference structure that rationalizes a choice correspondence on $\mathfrak{X}$, there is a transitive, but possibly incomplete, preference relation that rationalizes the same choice correspondence. (That is, for every transitive weak preference structure $(\succsim, \mathbf{R})$ on X, there is a preorder $\trianglerighteq$ on X such that $\bigcirc(\mathbf{MAX}(\cdot, \succsim), \mathbf{R}) = \mathbf{MAX}(\cdot, \trianglerighteq)$ on $\mathfrak{X}$.) This is a bit of a surprising observation, tying transitive weak preference structures (which abound in the literature) to incomplete preferences.

Theorem 5.3. *For any choice correspondence C on $\mathfrak{X}$, the following are equivalent*:

(a) *C satisfies Axioms* (B2), (B4) *and the Chernoff Axiom*;

(b) *C is rationalizable by a transitive weak preference structure*;

(c) *C is rationalizable by a* (*possibly incomplete*) *preference relation.*

We can in fact derive the (incomplete) preferences used in (c) from (b) explicitly. To wit, let C be a choice correspondence on $\mathfrak{X}$ rationalized by the transitive weak preference structure $(\succsim, \mathbf{R})$ on X. Then, we show while proving Theorem 5.3 in the Appendix that the C-dominance partial order rationalizes C, that is, $C(S)$ contains precisely the C-undominated elements in S. Moreover, given that $\mathbf{R}$ is transitive here, we can compute the C-dominance order as: x C-dominates y iff

$$\text{either } x \succ y \text{ or } [(x, y) \in \text{Inc}(\succsim) \text{ and } x\, \mathbf{R}^{>}\, y].$$

Thus, to compute $C(S)$, it is enough to identify the maximal elements in S with respect to this partial order.

Example. Consider the setting we described in Section 4.3 for our application to decision making under uncertainty, with a given set of priors Π, the Gilboa-Schmeidler maxmin expected utility function $\mathbf{R}_\Pi$, and Bewley preferences $\succsim_\Pi$ on the set of acts $\mathbb{R}^n$. In this context, the equivalence of (b) and (c) above means that for any finite menu of acts S, maximizing $\mathbf{R}_\Pi$ on $\mathbf{MAX}(S, \succsim_\Pi)$ is the same as computing $\mathbf{MAX}(S, \trianglerighteq_\Pi)$ where $\trianglerighteq_\Pi$ is the partial order on $\mathbb{R}^n$ defined as $x \vartriangleright_\Pi y$ iff

$$\text{either} \quad \begin{array}{l} \pi x \geq \pi y \text{ for all } \pi \in \Pi \\ \pi x > \pi y \text{ for some } \pi \in \Pi \end{array} \quad \text{or} \quad \begin{array}{l} \pi x > \pi y \text{ for some } \pi \in \Pi \\ \pi x < \pi y \text{ for some } \pi \in \Pi \\ \min_{\pi \in \Pi} \pi x > \min_{\pi \in \Pi} \pi y. \end{array}$$

As a side benefit of Theorem 5.3, we may readily conclude that $\trianglerighteq_\Pi$ is transitive without checking this explicitly. □

Remark. While empirical tests of choice models are conducted by means of finite choice problems, such models are often employed in economic contexts using infinite choice problems (as in the demand theory). It is thus worth noting which of the behavioral postulates above are obeyed by choice rules that are rationalized by preference structures in general. To this end, in this remark, let us agree to call a map $C : 2^X \backslash \{\varnothing\} \to 2^X$ a *choice correspondence* if $C(S) \subseteq S$ for every nonempty $S \subseteq X$, and $C(S) \neq \varnothing$ for every nonempty finite $S \subseteq X$. In turn, we say that C is *rationalized by a* (*transitive weak*) *preference structure* $(\succsim, \mathbf{R})$ *on X universally* if $C(S) = \bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R})$ for every nonempty $S \subseteq X$ with $C(S) \neq \varnothing$. Finally, let us refer to the version of (A1) in which S is assumed to be C-irreducible as (B1).

The proofs we give in Appendix can be modified to show that any choice correspondence C that is rationalized by a preference structure universally satisfies the properties (B1) and (B2), where we replace $\mathfrak{X}$ with $2^X \backslash \{\varnothing\}$ in their statements. However, the situation is a bit different for (B3) and (B4) (as well as the Chernoff Axiom in relation to rationalization by a transitive weak preference structure). First, one must confine these axioms to menus on which C is nonempty-valued. Second, to ensure the satisfaction of these properties in either Theorem 5.2 or 5.3 (with $2^X\backslash\{\varnothing\}$ playing the role of $\mathfrak{X}$), we need to impose a restriction on $\succsim$, along the lines of the Remarks in Section 4. In particular, it suffices to have a *weakly Noetherian* preorder $\succsim$ such that for every nonempty $S \subseteq X$ with $\mathbf{MAX}(S, \succsim) \neq \varnothing$ and every non-$\succsim$-maximal y in S, there exists a $\succsim$-maximal x in S with $x \succ y$. (In the Online Appendix, we develop an alternative topological approach focusing on compact menus in a metric space.)

Remark. The incomplete preferences derived in parts (b) and (c) of Theorem 5.3 need not be the same. In applications, for instance, one may start with a particular core preference relation (such as the Pareto order, or the first-order stochastic dominance, etc.), and choose a particular transitive superrelation of this core preferences. Clearly, applying the choice rule with these will not choose all maximal elements with respect to the core relation; it will often result in a vast refinement instead. What Theorem 5.3 says is that the totality of what are chosen in this way can also be viewed as the maximal set with respect to some other preorder (such as the C-dominance partial order).

Remark. The axioms (A1) and (B2)-(B4), as well as (B2), (B4) and the Chernoff Axiom, are logically independent. This is established by means of simple examples which, for brevity, we omit giving here.

6 On the Identification of Core Preferences

6.1 The Case of Preference Structures

Let C be a choice correspondence on X that satisfies the axioms (A1) and (B2)-(B4). Then, Theorem 5.2 says that there exists a preference structure $(\succsim, \mathbf{R})$ on X that rationalizes C. What can we say about this preference structure in terms of the choice behavior modeled by C? An immediate observation is that $\mathbf{R}$ is none other than what C reveals over pairwise choice problems. That is, where $\mathbf{R}_C$ is the binary relation on X defined by

$$x \, \mathbf{R}_C \, y \quad \text{iff} \quad x \in C\{x, y\},$$

we have $\mathbf{R} = \mathbf{R}_C$. The situation is a bit more complicated when it comes to the core preference relation $\succsim$, as this relation is not uniquely identified by C. There is, however, a natural candidate in this regard: the *C-dominance relation.* To be precise, let us consider the partial order $\succcurlyeq_C$ on X defined by

$$x \succcurlyeq_C y \quad \text{iff} \quad x = y \text{ or } x \text{ } C\text{-dominates } y.$$

The proof of Theorem 5.2 we give in the Appendix shows that $\succcurlyeq_C$ is a viable core preference relation (to be deduced from C) in that $(\succcurlyeq_C, \mathbf{R}_C)$ is a preference structure that rationalizes C. But this is not the only preorder that can be used in this manner. Indeed, Nishimura and Ok (2019) show that the collection of all preorders which, when paired with $\mathbf{R}_C$, yields a preference structure that rationalizes C is a complete lattice (with respect to the extension relation); the asymmetric part of the largest member of this lattice is precisely $\succ_C$. In this section, our objective is to characterize all members of this lattice, that is, determine all preorders $\succsim$ such that $(\succsim, \mathbf{R}_C)$ is a preference structure on X that rationalizes C.

Our characterization is based on the following auxiliary binary relation.

Definition. Let C be a choice correspondence on $\mathfrak{X}$. The binary relation $\blacktriangleright_C$ on X is defined as $x \blacktriangleright_C y$ iff there exist a $(z, S) \in X \times \mathfrak{X}$ such that

$$z \in C(S \cup \{x\}), \quad y \notin C(S \cup \{x\}), \quad y \in C\{y,z\} \quad \text{and} \quad y \in C(S). \tag{10}$$

The importance of this relation stems from the fact that if $(\succsim, \mathbf{R})$ is a preference structure on X that rationalizes C, then the strict core preference $\succ$ must contain $\blacktriangleright_C$. Indeed, where $C(S \cup \{x\}) = \bigcirc(\mathbf{MAX}(S \cup \{x\}, \succsim), \mathbf{R})$, the conditions $z \in C(S \cup \{x\})$ and $y \in C\{y,z\}$ jointly imply $y \in C(S \cup \{x\})$ whenever y is a $\succsim$-maximal element of S. So, if y does not belong to $C(S \cup \{x\})$, it must be the case that some element of $S \cup \{x\}$ dominates y with respect to $\succ$. But if y belongs to $C(S)$, it is $\succsim$-maximal in S, so the only way this is possible is when $x \succ y$. Thus: $x \blacktriangleright_C y$ implies $x \succ y$. Thus, while $\succsim$ is not observable (as it is not unique), a major section of its strict part, namely, $\blacktriangleright_C$, is apparently observable.

The following result complements Theorem 5.2 by telling us exactly which sorts of preference structures may rationalize a given choice correspondence.

Theorem 6.1. *Let $(\succsim, \mathbf{R})$ be a preference structure on X, and C a choice correspondence on $\mathfrak{X}$. Then, $(\succsim, \mathbf{R})$ rationalizes C if, and only if, the following conditions hold:*
(a) *C satisfies Axioms* (A1) *and* (B2)-(B4);
(b) $\mathbf{R} = \mathbf{R}_C$;
(c) $\succ_C \supseteq \succ \supseteq \blacktriangleright_C$.

The "if" part of this result says that a choice correspondence C that satisfies (A1) and (B2)-(B4) can be rationalized by a given preference structure $(\succsim, \mathbf{R})$ whenever $\mathbf{R} = \mathbf{R}_C$ and the strict part of $\succsim$ is contained within the order interval between $\blacktriangleright_C$ and $\succ_C$.[13] One may use this fact to test the descriptive power of more specific versions of our choice theory, or to formulate a more refined representation in terms of a particular class of core preferences.[14] As for the "only if" part of Theorem 6.1, condition (c) uncovers the extent of uniqueness of the strict part of core preferences that rationalize a given choice correspondence.[15]

6.2 The Case of Transitive Weak Preference Structures

We next determine the core part of a transitive weak preference structure $(\succsim, \mathbf{R})$ on X from the choices induced by such a structure. In this case, the condition (b) of Theorem 6.1 requires a modification because $x \mathbf{R} y$ no longer implies $x \mathbf{R}_C y$. Instead, we focus on the equation (9) for binary menus. In fact, if this equation holds for all doubleton S, then the added strength of the Chernoff Axiom makes checking (c) in Theorem 6.1 redundant. The remaining two conditions are modified in the following manner:

[13]In particular, it can be shown that $(\succcurlyeq_C, \mathbf{R}_C)$ and $(\text{tran}(\blacktriangleright_C) \cup \triangle_X, \mathbf{R}_C)$ are preference structures that rationalize C with the largest and smallest strict core preferences, respectively.

[14]Consider a certain class of preorders $\mathbb{P}$. If a choice correspondence C satisfies (A1) and (B2)-(B4), then in order to represent C with a core preference that belongs to $\mathbb{P}$, it would suffice to find a preorder $\succsim \in \mathbb{P}$ such that (i) $(\succsim, \mathbf{R}_C)$ is a preference structure, and (ii) condition (c) of Theorem 6.1 holds.

[15]In general, core indifferences are more loosely identified than the strict part of core preferences. But, in many contexts that are encountered in economic models, this is not likely to be a pressing issue. For instance, when X is a topological space, one may focus on closed preorders, and use approximation methods to recover the symmetric part of core preferences from their asymmetric part.

Theorem 6.2. *Let* $(\succsim, \mathbf{R})$ *be a transitive weak preference structure on* X*, and* C *a choice correspondence on* $\mathfrak{X}$*. Then,* $(\succsim, \mathbf{R})$ *rationalizes* C *if, and only if, the following two conditions hold:*

(a') *C satisfies Axioms* (B2), (B4) *and the Chernoff Axiom*;

(b') *For any* $x, y \in X$*, we have* $x \in C\{x, y\}$ *iff* $x \mathbf{R} y$ *and not* $y \succ x$.

It is plain that a choice correspondence C is rational in the classical sense, that is, its values are obtained by maximizing a complete and transitive binary relation $\mathbf{R}$, if and only if C satisfies Arrow's Choice Axiom, and $\mathbf{R} = \mathbf{R}_C$. Theorem 6.2 comes fairly close to this characterization: Arrow's Choice Axiom is replaced by three weaker properties, namely (B2), (B4) and the Chernoff Axiom, while the property $\mathbf{R} = \mathbf{R}_C$ is replaced by the condition $x \mathbf{R}_C y$ iff $x \mathbf{R} y$ *and* not $y \succ x$ (for any $x, y \in X$).

6.3 More on the Uniqueness of Preference Structures

Theorems 6.1 and 6.2 identify the extent to which one's choice behavior determines her rationalizing preference structure in the most general setup possible. In applications, however, one often considers preference structures that belong to a particular class, and this may yield sharper uniqueness properties. As a final order of business, we illustrate this by means of two examples in the context of decision making under uncertainty.

Example. Let π^1, π^2 and π^3 be affinely independent probability n-vectors, and put $\Pi := \{\pi^1, \pi^2, \pi^3\}$. We consider the preference structure $(\succsim_\Pi, \mathbf{R}_{\text{maj}})$ where $\succsim_\Pi$ is the Bewley preferences on $\mathbb{R}^n$ as defined in Section 4.3, while $\mathbf{R}_{\text{maj}}$ is the majority relation on $\mathbb{R}^n$ defined by $x \mathbf{R}_{\text{maj}} y$ iff $|\{i : \pi^i x > \pi^i y\}| \geq |\{i : \pi^i y > \pi^i x\}|$. The affine independence of π^1, π^2 and π^3 means that no member of Π can be expressed as a convex combination of the other two, and this ensures that there are Condorcet cycles in $\mathbb{R}^n$ with respect to $\mathbf{R}_{\text{maj}}$.[16]

Now take any preference structure $(\succsim, \mathbf{R})$ on $\mathbb{R}^n$ such that $\succsim$, just as the relation $\succsim_\Pi$, is a closed preorder that satisfies the *Independence Axiom*: $x \succsim y$ iff $\alpha x + z \succsim \alpha y + z$ for any $x, y, z \in \mathbb{R}^n$ and $\alpha > 0$. We claim that $(\succsim_\Pi, \mathbf{R}_{\text{maj}})$ and $(\succsim, \mathbf{R})$ rationalize the same choice correspondence *iff* they are the same structure. Thus, the choice correspondence induced by $(\succsim_\Pi, \mathbf{R}_{\text{maj}})$ identifies the core preference of its rationalizing preference structure *uniquely*, among all closed preorders on $\mathbb{R}^n$ that satisfy the Independence Axiom.

To see this, suppose $(\succsim_\Pi, \mathbf{R}_{\text{maj}})$ and $(\succsim, \mathbf{R})$ rationalize the same choice correspondence C on the collection of all nonempty finite subsets of $\mathbb{R}^n$. Then, $\mathbf{R}_{\text{maj}}$ and $\mathbf{R}$ are both equal to $\mathbf{R}_C$, as noted earlier. Pick any $y, z, w \in \mathbb{R}^n$ with $y \mathbf{R}^>_{\text{maj}} z \mathbf{R}^>_{\text{maj}} w \mathbf{R}^>_{\text{maj}} y$. As $\mathbf{R}_{\text{maj}} = \mathbf{R}$, and $\mathbf{R}$ is a $\succsim$-transitive superrelation of $\succsim$, no two elements of $S := \{y, z, w\}$ can be $\succsim$-comparable. This implies $C(S) = S$ because $(\succsim, \mathbf{R}_{\text{maj}})$ rationalizes C. Now take any n-vectors u and v with $u \succ v$, and note that $x(\varepsilon) := y + \varepsilon(u - v) \succ y$ for each $\varepsilon > 0$. By a routine continuity argument, we can pick ε small enough that $x \mathbf{R}^>_{\text{maj}} z \mathbf{R}^>_{\text{maj}} w \mathbf{R}^>_{\text{maj}} x$ where $x := x(\varepsilon)$. But then $C(S \cup \{x\}) = \{x, z, w\}$ because $x \succ y$. Given $y \mathbf{R}^>_{\text{maj}} z$, it

[16] The existence of such cycles is not obvious, but it can be proved by means of elementary convex analysis. More specifically, one can use the affine independence hypothesis and the Separating Hyperplane Theorem to show that there exist acts y, z and w in $\mathbb{R}^n$ such that $\pi^1 y > \pi^1 z > \pi^1 w$, $\pi^2 w > \pi^2 y > \pi^2 z$, and $\pi^3 z > \pi^3 w > \pi^3 y$, which jointly imply $y \mathbf{R}^>_{\text{maj}} z \mathbf{R}^>_{\text{maj}} w \mathbf{R}^>_{\text{maj}} y$.

follows that $x \blacktriangleright_C y$. So, by Theorem 6.1, $y + \varepsilon(u - v) = x \succ_\Pi y$, whence $u \succ_\Pi v$. Thus: $\succ_\Pi \supseteq \succ$. Besides, we can analogously show that $\succ_\Pi \subseteq \succ$ as well.[17] Finally, since $\succsim_\Pi$ and $\succsim$ are closed preorders with nonempty strict parts, the Independence Axiom implies that these preorders are equal to the closures of their strict parts. Thus: $\succsim_\Pi = \succsim$. □

Example. Let Π_1 and Π_2 be nonempty, closed and convex sets of probability n-vectors, and define $\succsim_{\Pi_1}$ and $\succsim_{\Pi_2}$, and $\mathbf{R}_{\Pi_1}$ and $\mathbf{R}_{\Pi_2}$ as in Section 4.3. Let $\mathfrak{X}$ stand for the collection of all nonempty finite subsets of $\mathbb{R}^n$, and suppose that the transitive weak preference structures $(\succsim_{\Pi_1}, \mathbf{R}_{\Pi_1})$ and $(\succsim_{\Pi_2}, \mathbf{R}_{\Pi_2})$ rationalize the same choice correspondence C on $\mathfrak{X}$. Then, in fact, $\Pi_1 = \Pi_2$, so these two preference structures are one and the same.

To prove this, suppose there is a probability vector π_2 in $\Pi_2 \backslash \Pi_1$. Then, by the Separating Hyperplane Theorem, there is an $(x, \alpha) \in \mathbb{R}^n \times \mathbb{R}$ with $\min_{\pi \in \Pi_1} \pi x > \alpha > \pi_2 x$. So, where a stands for the n-vector whose every term equals α, we have $x \succ_{\Pi_1} a$ and $a \ \mathbf{R}^{>}_{\Pi_2} \ x$. Theorem 6.2 then yields $C\{x, a\} = \{x\}$ (because $(\succsim_{\Pi_1}, \mathbf{R}_{\Pi_1})$ rationalizes C) and $C\{x, a\} = \{a\}$ (because $(\succsim_{\Pi_2}, \mathbf{R}_{\Pi_2})$ rationalizes C), a contradiction. As an analogous argument shows that $\Pi_1 \backslash \Pi_2$ is empty as well, we conclude: $\Pi_1 = \Pi_2$. □

7 Conclusion

In this paper we aimed at providing a behavioral foundation for a two-stage rational choice model. The model uses two relations; the first (core preferences) is transitive, and the second (revealed preferences) is a relatively transitive completion of the first. The first stage of the model determines the set of maxima with respect to the core preferences, and its second stage takes the top-cycle of the latter set with respect to the revealed preferences. As it generalizes the classical rational choice theory, this model has superior explanatory power. While the generality of the model raises the concern that its predictive power may be poor, we found that this is not the case. In fact, the model admits behavioral characterizations that closely parallel those of the classical theory. Our results also provide minimal sets of falsifiable properties for the testing of the model.

In standard choice theory, one can recover the preferences of a rational decision maker from her behavior simply by recording her choices over pairwise choice problems. Doing this for the present model yields only the second relation; this is why we refer to it as revealed preferences. Core preferences can only be identified up to an equivalence class. We have characterized this class fully in the body of the paper, that is, we determined the set of *all* core preferences that are "revealed" by a rationalizable choice correspondence. In general, one may need this entire set for performing comparative statics exercises. However, in the context of concrete economic environments, one may impose further rationality properties at the outset, and may even be able to totally elicit the (unobservable) core preferences from the associated (observable) choice behavior. We have demonstrated this possibility above by means of some applications to the theory of decision making under uncertainty.

[17] This fact uses only the continuity of the revealed preference relation $\mathbf{R}$, the existence of an $\mathbf{R}^{>}$-cycle, and the independence axiom on the part of core preferences. When X is a Euclidean space, combining Theorem 6.1 with these assumptions yield that the asymmetric part of the core preferences of a preference structure that rationalizes a given choice correspondence is *unique*.

Throughout the exposition, we focused on finite choice menus, but we also explained how our findings can be extended to the case of infinite menus. Following a topological approach, the online appendix of the present paper fully develops such an extension.

APPENDIX: Proofs

Proof of Proposition 4.1

Take any nonempty finite subset S of X, and to simplify the notation, write $M(S)$ for $\mathbf{MAX}(S, \succsim)$. We observe first that

$$\bigcirc(M(S), \mathbf{R}) \subseteq \bigcirc(S, \mathbf{R}). \tag{11}$$

Indeed, if $x \in \bigcirc(M(S), \mathbf{R})$, then Lemma 3.1 says that x tran$(\mathbf{R}|_{M(S)})$ $M(S)$. Moreover, for any $z \in S \backslash M(S)$, finiteness of S (and transitivity of $\succsim$) ensure that there is a $y \in M(S)$ with $y \succ z$, and hence y $\mathbf{R}$ z, and hence x tran$(\mathbf{R}|_S)$ z. It follows that x tran$(\mathbf{R}|_S)$ $S \backslash M(S)$ as well, that is, x tran$(\mathbf{R}|_S)$ S. By Lemma 3.1, therefore, (11) holds.

Next, we claim that

$$\bigcirc(M(S), \mathbf{R}) \subseteq M(\bigcirc(S, \mathbf{R})) \subseteq M(S). \tag{12}$$

(Here, again, by $M(\bigcirc(S, \mathbf{R}))$, we mean $\mathbf{MAX}(\bigcirc(S, \mathbf{R}), \succsim)$.) Indeed, if $x \in \bigcirc(M(S), \mathbf{R})$, (11) says that $x \in \bigcirc(S, \mathbf{R})$, so, if x were not to belong to $M(\bigcirc(S, \mathbf{R}))$, there would exist some $y \in \bigcirc(S, \mathbf{R})$ with $y \succ x$, which contradicts the $\succsim$-maximality of x in S. This proves the first part of (12). To prove its second part, take any x in $M(\bigcirc(S, \mathbf{R}))$, and suppose x is not $\succsim$-maximal in S so that $y \succ x$ for some $y \in S$. Since $x \in \bigcirc(S, \mathbf{R})$, then, y $\mathbf{R}$ x tran$(\mathbf{R}|_S)$ S, so by Lemma 3.1, $y \in \bigcirc(S, \mathbf{R})$, but this contradicts $\succsim$-maximality of x in $\bigcirc(S, \mathbf{R})$.

Now, by definition, $\bigcirc(M(S), \mathbf{R})$ is an $\mathbf{R}$-highset in $M(S)$. It then follows from (12) that $\bigcirc(M(S), \mathbf{R})$ is an $\mathbf{R}$-highset in $M(\bigcirc(S, \mathbf{R}))$ as well. Since $\bigcirc(M(S), \mathbf{R})$ is obviously an $\mathbf{R}$-cycle, Lemma 3.1 says that it must be the top-cycle in $M(\bigcirc(S, \mathbf{R}))$ with respect to $\mathbf{R}$, as we sought.

Proof of Proposition 4.2

The following result shows that, in the context of a weak preference structure $(\succsim, \mathbf{R})$, the asymmetric part of $\mathbf{R}$ is also $\succsim$-transitive.

Lemma A.1. *Let $(\succsim, \mathbf{R})$ be a weak preference structure on a nonempty set X. Then,*

$$x \succsim y \ \mathbf{R}^> \ z \text{ (or } x \ \mathbf{R}^> \ y \succsim z) \quad \text{implies} \quad x \ \mathbf{R}^> \ z$$

for every $x, y, z \in X$.

Proof. Take any $x, y, z \in X$ with $x \succsim y$ $\mathbf{R}^>$ z but assume that x $\mathbf{R}^>$ z is false. As $\mathbf{R}$ is complete, we then have z $\mathbf{R}$ x. So, z $\mathbf{R}$ $x \succsim y$ and we find z $\mathbf{R}$ y (by $\succsim$-transitivity of $\mathbf{R}$) contradicting y $\mathbf{R}^>$ z. Analogously, x $\mathbf{R}^>$ $y \succsim z$ implies x $\mathbf{R}^>$ z as well. ■

We now turn to the proof of Proposition 4.2. Let S be a nonempty finite subset of X, write $M(S)$ for $\mathbf{MAX}(S, \succsim)$, and take any y in S that does not belong to $C(S) = \bigcirc(M(S), \mathbf{R})$. If $y \in M(S)$, then $C(S)$ $\mathbf{R}^>$ y, because $C(S)$ is an $\mathbf{R}$-highset in $M(S)$. Assume, then, $z \succ y$ for some $z \in S$. Since S is finite, this implies $x \succ y$ for some $x \in M(S)$. If x belongs to $C(S)$, we are done, so assume instead that x is in $M(S) \backslash C(S)$. Then, $C(S)$ $\mathbf{R}^>$ x because $C(S)$ is an $\mathbf{R}$-highset in $M(S)$. Thus, $C(S)$ $\mathbf{R}^>$ $x \succ y$, so, by Lemma A.1, we find $C(S)$ $\mathbf{R}^>$ y, as we sought.

Proof of Theorem 5.1

Assume that C satisfies (A1) and (A2), and define $\mathbf{R} \subseteq X \times X$ by $x\ \mathbf{R}\ y$ iff $x \in C\{x, y\}$. It is plain that $\mathbf{R}$ is total (because C is nonempty-valued), and $x\ \mathbf{R}^{>}\ y$ iff $\{x\} = C\{x, y\}$. Let us fix an arbitrary $S \in \mathfrak{X}$. Note that if $y\ \mathbf{R}\ x$ holds for some $x \in C(S)$ and $y \in S \backslash C(S)$, then $C(S) \cap \{x, y\} \neq \varnothing$, so (A2) dictates that $y \in C\{x, y\} \subseteq C(S) \cap \{x, y\}$, contradicting y lying outside of $C(S)$. Conclusion: $x\ \mathbf{R}^{>}\ y$ for every $x \in C(S)$ and $y \in S \backslash C(S)$, that is, $C(S)$ is an $\mathbf{R}$-highset in S. On the other hand, let A be an $\mathbf{R}$-highset in S . Then, $x\,\mathbf{R}^{>}\,y$, that is, $\{x\} = C\{x, y\}$, for every $x \in A$ and $y \in S \backslash A$. Hence, by (A1), $C(S) = C(A \cup (S \backslash A)) \subseteq A$. Conclusion: $C(S)$ is the smallest $\mathbf{R}$-highset in S. That is, $C(S) = \bigcirc(S, \mathbf{R})$, as we sought.

Conversely, assume that $C(S) = \bigcirc(S, \mathbf{R})$ for every $S \in \mathfrak{X}$. Let us first show that C satisfies (A1). To this end, take any $S, T \in \mathfrak{X}$ such that $C\{x, y\} = \{x\}$ for every $x \in S$ and $y \in T$. Then, by the representation of C, we have $x\ \mathbf{R}^{>}\ y$ for every $x \in S$ and $y \in T \backslash S$. In other words, S is an $\mathbf{R}$-highset in $S \cup T$. Since $C(S \cup T)$ is the smallest $\mathbf{R}$-highset in $S \cup T$ (by the representation of C), it follows that $C(S \cup T) \subseteq S$, as we sought.

It remains to show that C satisfies (A2). To this end, take any $S, T \in \mathfrak{X}$ with $S \subseteq T$ and $C(T) \cap S \neq \varnothing$. Let x be an element of $C(S)$, and pick any y in $C(T) \cap S$. Then, $y \in \bigcirc(T, \mathbf{R})$, so, Lemma 3.1 implies that, for every $z \in T$, there exist finitely many $w_0, ..., w_k \in T$ with $y = w_0\ \mathbf{R} \cdots \mathbf{R}\ w_k = z$. But since $y \in S$ and $x \in \bigcirc(S, \mathbf{R})$, applying Lemma 3.1 again, we find finitely many $w_0', ..., w_l' \in S$ with $x = w_0'\ \mathbf{R} \cdots \mathbf{R}\ w_l' = y$. As $S \subseteq T$, these two findings combine to show that $x\ \mathrm{tran}(\mathbf{R}|_T)\ z$ for every $z \in T$, that is, $x \in \max(T, \mathrm{tran}(\mathbf{R}|_T))$. Applying Lemma 3.1 one more time, then, we get $x \in \bigcirc(T, \mathbf{R}) = C(T)$. In view of the arbitrary choice of x, we thus find that $C(S) \subseteq C(T) \cap S$, and our proof is complete.

Proof of Theorem 5.2

We begin with a useful lemma.

Lemma A.2. *Let $(\succsim, \mathbf{R})$ be a weak preference structure on a nonempty set X. Define a binary relation $\mathbf{R}_{\succsim}$ on X as follows: $x\,\mathbf{R}_{\succsim}\,y$ iff*

$$\text{either } x \succsim y \text{ or } [(x, y) \in \mathrm{Inc}(\succsim) \text{ and } x\ \mathbf{R}\ y].$$

Then, $(\succsim, \mathbf{R}_{\succsim})$ is a preference structure on X.

Proof. That $\mathbf{R}_{\succsim}$ is a completion of $\succsim$ follows readily from its definition, so we need only to prove that $\mathbf{R}_{\succsim}$ is $\succsim$-transitive. To this end, take any x, y and z in X such that $x\ \mathbf{R}_{\succsim}\ y \succsim z$. Notice that $z \succ x$ cannot hold, because otherwise $y \succ x$ (by transitivity of $\succsim$), and hence $y\ (\mathbf{R}_{\succsim})^{>}\ x$ (because $\mathbf{R}_{\succsim}$ is an extension of $\succsim$), a contradiction. Thus: Either $x \succsim z$ or $(x, z) \in \mathrm{Inc}(\succsim)$. In the former case, we have $x\ \mathbf{R}_{\succsim}\ z$ by definition of $\mathbf{R}_{\succsim}$, so we are done. Suppose now that $(x, z) \in \mathrm{Inc}(\succsim)$. Since $\mathbf{R}$ contains $\succsim$, from the definition of $\mathbf{R}_{\succsim}$ it easily follows that $\mathbf{R}$ also contains $\mathbf{R}_{\succsim}$. Thus, $x\ \mathbf{R}_{\succsim}\ y$ implies $x\ \mathbf{R}\ y$. Then, $x\ \mathbf{R}\ y \succsim z$, and hence $x\ \mathbf{R}\ z$ by $\succsim$-transitivity of $\mathbf{R}$. So, $(x, z) \in \mathrm{Inc}(\succsim)$ and $x\ \mathbf{R}\ z$, which jointly imply $x\ \mathbf{R}_{\succsim}\ z$. One can similarly show that $x \succsim y\ \mathbf{R}_{\succsim}\ z$ implies $x\ \mathbf{R}_{\succsim}\ z$. Conclusion: $\mathbf{R}_{\succsim}$ is $\succsim$-transitive, and $(\succsim, \mathbf{R}_{\succsim})$ is a preference structure on X. ■

We now proceed to prove Theorem 5.2.

(a) ⇒ (b). Consider the binary relation $\succcurlyeq_C$ on X defined by

$$x \succcurlyeq_C y \quad \text{iff} \quad x = y \text{ or } x\ C\text{-dominates } y.$$

We claim that this relation is transitive. (As it is obviously antisymmetric, this will establish that $\succcurlyeq_C$ is a partial order on X.) To see this, take any $x, y, z \in X$ with $x \succcurlyeq_C y \succcurlyeq_C z$. If any two of these alternatives are the same, we trivially have $x \succcurlyeq_C z$. So, let us assume this is not the case so that $x \succ_C y \succ_C z$. Take any $S \in \mathfrak{X}$ with $x, z \in S$. Since $x \succ_C y$, the set $C(S \cup \{y\})$ does not contain y, that is, $C(S \cup \{y\}) \subseteq S$. By (B4), therefore, $C(S \cup \{y\}) = C(S)$. But then $C(S)$ does not contain z, as we seek, because $y \succ_C z$ implies that z does not belong to $C(S \cup \{y\})$. In view of the arbitrary choice of S, we conclude that x C-dominates z, or equivalently, $x \succ_C z$. Thus, $\succcurlyeq_C$ is transitive as we claimed.

We next define $\mathbf{R} \subseteq X \times X$ by $x \mathbf{R} y$ iff $x \in C\{x, y\}$. It is plain that $\mathbf{R}$ is total, $x \mathbf{R}^> y$ iff $\{x\} = C\{x, y\}$ and $x \neq y$, while $x \mathbf{R}^= y$ iff $\{x, y\} = C\{x, y\}$. Whenever $x \succ_C y$, the definition of $\succcurlyeq_C$ combined with the characterization of $\mathbf{R}^>$ that we just noted ensure that $x \mathbf{R}^> y$. Given that $\succcurlyeq_C$ is a partial order, therefore, we conclude that $\mathbf{R}$ is a completion of $\succcurlyeq_C$.

Our next task is to show that $\mathbf{R}$ is $\succcurlyeq_C$-transitive. To this end, take any $x, y, z \in X$ with $x \succcurlyeq_C y \mathbf{R} z$. Let us assume that these alternatives are pairwise distinct, for things are trivial otherwise. Then $x \succ_C y$, and hence, $C\{x, y, z\} \subseteq \{x, z\}$, while (B4) implies $C\{x, y, z\} = C\{x, z\}$. Now, to derive a contradiction, suppose $z \mathbf{R}^> x$, so that $C\{x, z\} = \{z\} \subseteq \{y, z\}$. This implies $C\{x, y, z\} \subseteq \{y, z\}$ because $C\{x, y, z\} = C\{x, z\}$, as noted earlier. We may thus apply (B4) to get $C\{x, y, z\} = C\{y, z\}$. Since, by our earlier observations, we also have $C\{x, y, z\} = \{z\}$, it follows that $C\{y, z\} = \{z\}$, which contradicts the hypothesis $y \mathbf{R} z$. Hence, $x \mathbf{R} z$ holds, as we sought.

Suppose now that $x \mathbf{R} y \succcurlyeq_C z$. As before, we may assume that these are pairwise distinct alternatives. Then $y \succ_C z$, and reasoning as in the previous paragraph, we find $C\{x, y\} = C\{x, y, z\}$. Thus, $x \mathbf{R} y$ implies $x \in C\{x, y, z\}$. Since $C\{x, y, z\}$ contains at most two elements, (B3) applies to yield $x \in C\{x, z\}$, that is, $x \mathbf{R} z$, as we sought. Conclusion: $\mathbf{R}$ is $\succcurlyeq_C$-transitive.

We now know that $(\succcurlyeq_C, \mathbf{R})$ is a preference structure on X. It remains to prove that this preference structure rationalizes C. To this end, let us pick an arbitrary $S \in \mathfrak{X}$, and let us agree to write $M(S)$ for $\mathbf{MAX}(S, \succcurlyeq_C)$. Note first that if $x \in C(S)$, then $y \succ_C x$ holds for no $y \in S$. Put differently, $C(S) \subseteq M(S)$. It then follows from (B4) that $C(S) = C(M(S))$.

By definition of $\succcurlyeq_C$, the set $M(S)$ is C-irreducible. So, by (B2), $C\{x, y\} \subseteq C(M(S)) \cap \{x, y\}$ for any $(x, y) \in M(S) \times C(M(S))$. In other words, we have $x \in C(M(S))$ for every $x \in M(S)$ such that $x \mathbf{R} y$ for some $y \in C(M(S))$. Conclusion: $C(M(S))$ is an $\mathbf{R}$-highset in $M(S)$.

Pick any $\mathbf{R}$-highset T in $M(S)$. Since $M(S)$ is a C-irreducible set, no two distinct members of the subset T are $\succcurlyeq_C$-comparable, which means that T is also C-irreducible. Moreover, by definition of an $\mathbf{R}$-highset, we have $C\{x, y\} = \{x\}$ for every $x \in T$ and $y \in M(S) \backslash T$, and hence, (A1) implies $C(M(S)) \subseteq T$. (This inclusion holds trivially whenever $M(S) \backslash T$ is empty.) As T is arbitrarily selected, we have thereby shown that $C(M(S))$ is the smallest $\mathbf{R}$-highset in $M(S)$. By Lemma 3.1, this means $C(M(S)) = \bigcirc(M(S), \mathbf{R})$. Since $C(S) = C(M(S))$, it also follows that $C(S) = \bigcirc(M(S), \mathbf{R})$. Hence, $(\succcurlyeq_C, \mathbf{R})$ rationalizes C, as we claimed.

(b) $\Leftrightarrow$ (c). That (b) implies (c) is obvious. To prove the converse, assume that C is rationalized by a weak preference structure $(\succsim, \mathbf{R})$. By Lemma A.2, $(\succsim, \mathbf{R}_{\succsim})$ is a preference structure on X. We will now show that $(\succsim, \mathbf{R}_{\succsim})$ also rationalizes C. To this end, take any $S \in \mathfrak{X}$. As noted in the proof of Lemma A.2, $\mathbf{R}$ is a superrelation of $\mathbf{R}_{\succsim}$. Moreover, given any $\succsim$-maximal elements x and y of S, either we have $x \sim y$, and hence $x \mathbf{R}_{\succsim}^= y$, or else $(x, y) \in \text{Inc}(\succsim)$. In the latter case, $x \mathbf{R} y$ implies $x \mathbf{R}_{\succsim} y$ by the definition of $\mathbf{R}_{\succsim}$. Thus, for any $x, y \in \mathbf{MAX}(S, \succsim)$ with $x \mathbf{R} y$, we have $x \mathbf{R}_{\succsim} y$. It follows that the restrictions of $\mathbf{R}$ and $\mathbf{R}_{\succsim}$ to $\mathbf{MAX}(S, \succsim)$ are the same. Then, the top-cycles of these binary relations in $\mathbf{MAX}(S, \succsim)$ are also the same, meaning $\bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}) = \bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}_{\succsim})$. So, $C(S) = \bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R})$ implies $C(S) = \bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}_{\succsim})$. That is, rationalization by $(\succsim, \mathbf{R})$ implies rationalization by $(\succsim, \mathbf{R}_{\succsim})$.

(b) ⇒ (a). Let $(\succsim, \mathbf{R})$ be a preference structure on X that rationalizes C. Note that, for any x and y in X, we have $x \ \mathbf{R} \ y$ iff $x \in C\{x,y\}$. Moreover, if x and y are distinct, then $x \ \mathbf{R}^{>} \ y$ iff $\{x\} = C\{x,y\}$. The following fact is also worth noting:

Claim 1. $S = \mathbf{MAX}(S, \succsim)$ for every C-irreducible $S \in \mathfrak{X}$.

Proof of Claim 1. Take any $S \in \mathfrak{X}$ and suppose $S \backslash \mathbf{MAX}(S, \succsim) \neq \varnothing$. Then, for any x in the latter set, we have $y \succ x$ for some $y \in S$. By the representation of C, therefore, x does not belong to $C(T)$ for any $T \in \mathfrak{X}$ with $y \in T$, that is, y C-dominates x. Thus: S is not C-irreducible. ‖

Take any $S, T \in \mathfrak{X}$ such that $C\{x,y\} = \{x\}$ for every $x \in S$ and $y \in T$. Then, $x \ \mathbf{R}^{>} \ y$ for all $x \in S$ and $y \in T \backslash S$. Note that $y \succsim x$ cannot hold for such x and y because $\succsim \subseteq \mathbf{R}$. Consequently, $\mathbf{MAX}(S, \succsim) \subseteq \mathbf{MAX}(S \cup T, \succsim)$, and the set $S' := S \cap \mathbf{MAX}(S \cup T, \succsim)$ equals $\mathbf{MAX}(S, \succsim)$. Since $\mathbf{MAX}(S \cup T, \succsim) \backslash S'$ is a subset of $T \backslash S$, we also have $x \ \mathbf{R}^{>} \ y$ for all $x \in S'$ and $y \in \mathbf{MAX}(S \cup T, \succsim) \backslash S'$. It follows that S' is an $\mathbf{R}$-highset in $\mathbf{MAX}(S \cup T, \succsim)$. As $C(S \cup T)$ is the smallest $\mathbf{R}$-highset in $\mathbf{MAX}(S \cup T, \succsim)$, we thus find $C(S \cup T) \subseteq S' \subseteq S$. Thus: C satisfies (A1).

To prove (B2), pick any C-irreducible $S, T \in \mathfrak{X}$ with $S \subseteq T$ and $C(T) \cap S \neq \varnothing$. By Claim 1, we have $S = \mathbf{MAX}(S, \succsim)$ and $T = \mathbf{MAX}(T, \succsim)$. Now, let x be any element of $C(S)$, and pick a y in $C(T) \cap S$. Then, $y \in \bigcirc(T, \mathbf{R})$, so, Lemma 3.1 implies $y \ \mathrm{tran}(\mathbf{R}|_T) \ T$. But since $y \in S$ and $x \in \bigcirc(S, \mathbf{R})$, applying Lemma 3.1 again, we find $x \ \mathrm{tran}(\mathbf{R}|_S) \ y$ and, as $S \subseteq T$, $x \ \mathrm{tran}(\mathbf{R}|_T) \ y$. Hence, $x \ \mathrm{tran}(\mathbf{R}|_T) \ T$. Again by Lemma 3.1, we conclude that $x \in \bigcirc(T, \mathbf{R}) = C(T)$. Given the arbitrary choice of x, this proves that $C(S) \subseteq C(T) \cap S$, as demanded by (B2).

Take any $S, T \in \mathfrak{X}$ with $S \subseteq T$ and $|C(T)| \leq 2$. Then, $x \ \mathbf{R}^{=} \ y$ for any $x, y \in C(T)$, because $C(T)$ is an $\mathbf{R}$-cycle with at most two elements. As $C(T)$ is an $\mathbf{R}$-highset in $\mathbf{MAX}(T, \succsim)$, it follows that $x \ \mathbf{R} \ z$ for every $x \in C(T)$ and $z \in \mathbf{MAX}(T, \succsim)$. Moreover, since T is finite, for every $w \in T$ there is a $z \in \mathbf{MAX}(T, \succsim)$ with $z \succsim w$. Since $\mathbf{R}$ is $\succsim$-transitive, therefore, we find $x \ \mathbf{R} \ T$ for every $x \in C(T)$. In particular, as $C(S) \subseteq S \subseteq T$, we have

$$x \ \mathbf{R} \ C(S) \quad \text{for every } x \in C(T) \cap S. \tag{13}$$

Note also that $C(T) \cap S \subseteq \mathbf{MAX}(T, \succsim) \cap S \subseteq \mathbf{MAX}(S, \succsim)$. Now take any $x \in C(T) \cap S$. Then, $x \in \mathbf{MAX}(S, \succsim)$, as we just noted, while (13) implies $x \ \mathbf{R} \ C(S)$. We thus have $x \in C(S)$, because $C(S)$ is an $\mathbf{R}$-highset in $\mathbf{MAX}(S, \succsim)$. Thus, $C(T) \cap S \subseteq C(S)$, which proves (B3).

Finally, take any $S, T \in \mathfrak{X}$ with $S \subseteq T$ and $C(T) \subseteq S$. Then, $C(T) \subseteq \mathbf{MAX}(T, \succsim) \cap S \subseteq \mathbf{MAX}(S, \succsim)$. In fact, $C(T)$ is an $\mathbf{R}$-highset in $\mathbf{MAX}(S, \succsim)$. To see this, take any $x \in C(T)$ and $y \in \mathbf{MAX}(S, \succsim) \backslash C(T)$. If $y \in \mathbf{MAX}(T, \succsim)$, then $x \ \mathbf{R}^{>} \ y$, as we seek, because $C(T)$ is an $\mathbf{R}$-highset in $\mathbf{MAX}(T, \succsim)$. Thus, we may assume that y does not belong to $\mathbf{MAX}(T, \succsim)$. As T is finite, then, there exists a $z \in \mathbf{MAX}(T, \succsim)$ with $z \succ y$. Clearly, z does not belong to $C(T)$, because otherwise $z \in S$ (since $C(T) \subseteq S$), and this contradicts y being $\succsim$-maximal in S. Given that $C(T)$ is an $\mathbf{R}$-highset in $\mathbf{MAX}(T, \succsim)$ and $x \in C(T)$, therefore, $x \ \mathbf{R}^{>} \ z$. Since we also have $z \succ y$, Lemma A.1 then implies $x \ \mathbf{R}^{>} \ y$. In view of the arbitrary choice of x and y, we conclude that $C(T)$ is, indeed, an $\mathbf{R}$-highset in $\mathbf{MAX}(S, \succsim)$. As $C(T)$ is also an $\mathbf{R}$-cycle, from Lemma 3.1 it follows that $C(T) = \bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R})$, i.e., $C(T) = C(S)$. Conclusion: C satisfies (B4).

Proof of Theorem 5.3

If (c) holds, that is, there exists a preorder $\succsim$ on X such that $C(S) = \mathbf{MAX}(S, \succsim)$ for every $S \in \mathfrak{X}$, setting $\mathbf{R} := X \times X$ yields a transitive weak preference structure $(\succsim, \mathbf{R})$ on X that rationalizes C. Thus: (c) implies (b). We will complete our proof by showing that (a) implies (c), and (b) implies (a).

(a) ⇒ (c). Assume that C satisfies (B2), (B4) and the Chernoff Axiom. Define $\succcurlyeq_C$ and $\mathbf{R}$ as in the proof of the "(a) ⇒ (b)" part of Theorem 5.2. We have seen during the course of that proof that $(\succcurlyeq_C, \mathbf{R})$ is a preference structure on X such that

$$C(S) = \bigcirc(\mathbf{MAX}(S, \succcurlyeq_C), \mathbf{R}) \quad \text{for every } S \in \mathfrak{X}. \tag{14}$$

Claim 1. $\succ_C = \mathbf{R}^{>}$.

Proof of Claim 1. Since $(\succcurlyeq_C, \mathbf{R})$ is a preference structure, we have $\succ_C \subseteq \mathbf{R}^{>}$. Conversely, if $x \succ_C y$ is false, there exists an $S \in \mathfrak{X}$ such that $x \in S$ and $y \in C(S)$. But by the Chernoff Axiom, this implies $y \in C\{x, y\}$, which means that $x \; \mathbf{R}^{>} \; y$ is false. Conclusion: $\succ_C \supseteq \mathbf{R}^{>}$. ‖

Claim 2. $x \; \mathbf{R}^{=} \; y$ for every $x, y \in \mathbf{MAX}(S, \succcurlyeq_C)$ and $S \in \mathfrak{X}$.

Proof of Claim 2. Take any $S \in \mathfrak{X}$, and any $x, y \in \mathbf{MAX}(S, \succcurlyeq_C)$. Then, either $x = y$ or $(x, y) \in \mathrm{Inc}(\succcurlyeq_C)$. In the former case we obviously have $x \; \mathbf{R}^{=} \; y$ (because $\mathbf{R}$ is reflexive), and in the latter case, we have $x \; \mathbf{R}^{=} \; y$ because $\succ_C = \mathbf{R}^{>}$ (Claim 1) and $\mathbf{R}$ is complete. ‖

Claim 2 entails that $\bigcirc(\mathbf{MAX}(S, \succcurlyeq_C), \mathbf{R}) = \mathbf{MAX}(S, \succcurlyeq_C)$ for each $S \in \mathfrak{X}$. Combining this with (14), we find that C is rationalized by the partial order $\succcurlyeq_C$.

(b) $\Rightarrow$ (a). Let $C = \max(\mathbf{MAX}(\cdot, \succsim), \mathbf{R})$ for a transitive weak preference structure $(\succsim, \mathbf{R})$ on X. By Theorem 5.2, C satisfies (B2) and (B4), so we only need to show here that C satisfies the Chernoff Axiom. To this end, take any $S, T \in \mathfrak{X}$ with $S \subseteq T$, and let x be an arbitrary element of $C(T) \cap S$. As $C(T) \cap S \subseteq \mathbf{MAX}(T, \succsim) \cap S \subseteq \mathbf{MAX}(S, \succsim)$, we have $x \in \mathbf{MAX}(S, \succsim)$. To derive a contradiction, suppose $y \; \mathbf{R}^{>} \; x$ for some $y \in \mathbf{MAX}(S, \succsim)$. Note that either $y \in \mathbf{MAX}(T, \succsim)$ or, as T is finite, there is a $z \in \mathbf{MAX}(T, \succsim)$ with $z \succ y$. As $z \succ y \; \mathbf{R}^{>} \; x$ implies $z \; \mathbf{R}^{>} \; x$ (Lemma A.1), we may conclude that there is an element in $\mathbf{MAX}(T, \succsim)$ that is ranked by $\mathbf{R}$ strictly above x, but this contradicts $x \in C(T)$. Thus, $x \; \mathbf{R} \; y$ for all $y \in \mathbf{MAX}(S, \succsim)$, that is, $x \in C(S)$. Conclusion: C satisfies the Chernoff Axiom.

Proof of Theorem 6.1

We begin by defining the binary relation $\rhd_C$ on X by $x \rhd_C y$ iff there exist finitely many $z_1, ..., z_k \in X$ such that

(i) $y \; \mathbf{R}_C \; z_1 \; \mathbf{R}_C \cdots \mathbf{R}_C \; z_k \; \mathbf{R}_C \; x$;

(ii) $y \in C\{y, z_1, ..., z_k\}$;

(iii) $x, z_1, ..., z_k \in C(T)$ for some $T \in \mathfrak{X}$ with $y \in T$; and

(iv) $y \notin C\{x, y, z_1, ..., z_k\}$.

The following shows that this relation is contained within $\blacktriangleright_C$ whenever C is rationalizable.

Lemma A.3. *Assume that C is rationalizable by a preference structure. Then, $\rhd_C \subseteq \blacktriangleright_C$.*

Proof. Take any $x, y \in X$ with $x \rhd_C y$ so that there exist finitely many $z_1, ..., z_k \in X$ such that conditions (i)-(iv) above hold. Put $S := \{y, z_1, ..., z_k\}$, and $z := z_1$. Then, obviously, $y \in C(S)$ and $y \notin C(S \cup \{x\})$, while we have $y \in C\{z, y\}$ because $y \; \mathbf{R}_C \; z_1$. It remains to show that $z_1 \in C(S \cup \{x\})$. To this end, let $(\succsim, \mathbf{R})$ be a preference structure that rationalizes C, which entails $\mathbf{R} = \mathbf{R}_C$. We claim:

$$\mathbf{MAX}(S \cup \{x\}, \succsim) = \{x, z_1, ..., z_k\}. \tag{15}$$

By condition (iii) above, there is a $T \in \mathfrak{X}$ with $y \in T$ and $x, z_1, ..., z_k \in C(T)$. Then, x and $z_1, ..., z_k$ are all $\succsim$-maximal in T (by the representation of C), so as $S \cup \{x\} \subseteq T$, we see that $x, z_1, ..., z_k \in \mathbf{MAX}(S \cup \{x\}, \succsim)$. Moreover, if y were also $\succsim$-maximal in $S \cup \{x\}$, that is, $\mathbf{MAX}(S \cup \{x\}, \succsim) = \{x, y, z_1, ..., z_k\}$, condition (i) and Lemma 3.1 would have given $y \in C(S \cup \{x\})$, contradicting condition (iv). Thus, y is not $\succsim$-maximal in $S \cup \{x\}$, and hence (15) holds. But then

$$\begin{aligned} z_1 &\in \max\left(\mathbf{MAX}(S \cup \{x\}, \succsim), \text{tran}(\mathbf{R}|_{\mathbf{MAX}(S\cup\{x\},\succsim)})\right) \\ &= \bigcirc(\mathbf{MAX}(S \cup \{x\}, \succsim), \mathbf{R}) \\ &= C(S \cup \{x\}), \end{aligned}$$

where the first observation follows from (15) and condition (i), the second from Lemma 3.1, and the third from the representation of C. ■

We now extend $\rhd_C$ into a preorder as $\unrhd_C := \text{tran}(\rhd_C) \cup \triangle_X$. Coupling this relation with $\mathbf{R}_C$ results in a preference structure that rationalizes C, provided that C is rationalizable.

Lemma A.4. *Assume that C is rationalizable by a preference structure. Then, $(\unrhd_C, \mathbf{R}_C)$ is a preference structure that rationalizes C.*

Proof. See the Online Appendix of Nishimura and Ok (2019). ■

We are now in a position to prove Theorem 6.1. Suppose first that $(\succsim, \mathbf{R})$ rationalizes C. Then, (a) follows from Theorem 5.2, and that $\mathbf{R} = \mathbf{R}_C$ and $\succ_C \supseteq \succ$ follow readily from the representation of C. As we have already shown in Section 6 that $\succ \supseteq \blacktriangleright_C$ holds, the "only if" part of Theorem 6.1 is proved.

To prove the "if" part, suppose $(\succsim, \mathbf{R})$ and C satisfy the conditions (a)-(c) of the theorem. By Theorem 5.2, then, C is rationalizable by a preference structure, so, by Lemma A.4, $(\unrhd_C, \mathbf{R}_C)$ is a preference structure on X that rationalizes C.

Now take an arbitrary S in $\mathfrak{X}$. We claim:

$$\mathbf{MAX}(\mathbf{MAX}(S, \succsim), \unrhd_C) = \mathbf{MAX}(S, \succsim). \tag{16}$$

Indeed, we have

$$\begin{matrix}\text{the asymmetric} \\ \text{part of } \unrhd_C\end{matrix} \subseteq \text{tran}(\rhd_C) \subseteq \text{tran}(\blacktriangleright_C) \subseteq \ \succ .$$

(Here the first containment follows from the definition of $\unrhd_C$, the second from Lemma A.3, and the third from condition (c) and transitivity of $\succ$.) Thus, any element of S that does not belong to the left-hand side of (16) cannot be $\succsim$-maximal in S, which proves $\supseteq$ part of (16). The converse containment in this equation is trivially true.

Note also that for any $x \in C(S)$, there does not exist a $y \in S$ with $y \succ_C x$. Thus, the containment $\succ_C \supseteq \succ$ in condition (c) implies $C(S) \subseteq \mathbf{MAX}(S, \succsim)$. As $\mathbf{MAX}(S, \succsim) \subseteq S$ and C satisfies (B4), we find

$$C(S) = C(\mathbf{MAX}(S, \succsim)) = \bigcirc(\mathbf{MAX}(\mathbf{MAX}(S, \succsim), \unrhd_C), \mathbf{R}_C) = \bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}).$$

(Here the second equality follows because $(\unrhd_C, \mathbf{R}_C)$ rationalizes C, and the third from (16) and condition (b).) In view of the arbitrary choice of S, our proof is complete.

Proof of Theorem 6.2

Assume first that $(\succsim, \mathbf{R})$ rationalizes C. Then, (a') follows from Theorem 5.3, while we have (9), and hence $C\{x, y\} = \max(\{x, y\}, \mathbf{R}) \cap \mathbf{MAX}(\{x, y\}, \succsim)$ for any $x, y \in X$. One readily checks that this equality is equivalent to the condition (b').

Conversely, suppose $(\succsim, \mathbf{R})$ and C satisfy the conditions (a') and (b'). Take an arbitrary S in $\mathfrak{X}$. By the Chernoff Axiom, we have $x\ \mathbf{R}_C\ S$ for any $x \in C(S)$. Consequently, property (b') implies $x\ \mathbf{R}\ y$ and not $y \succ x$, for any $(x, y) \in C(S) \times S$. Conclusion: $C(S) \subseteq \max(S, \mathbf{R}) \cap \mathbf{MAX}(S, \succsim)$. To prove the converse containment, pick any $z \in \max(S, \mathbf{R}) \cap \mathbf{MAX}(S, \succsim)$. Then, obviously, $z \in \max(\{z, y\}, \mathbf{R}) \cap \mathbf{MAX}(\{z, y\}, \succsim)$ for every $y \in S$. By property (b'), then, $z\ \mathbf{R}_C$ S. Moreover, by (a') and Theorem 5.2, there is a preference structure $(\succsim', \mathbf{R}_C)$ that rationalizes C. Then, C only selects $\succsim'$-maximal elements from any binary menu. Since we have $z\ \mathbf{R}_C\ S$, it follows that there does not exist a $y \in S$ with $y \succ' z$, that is, $z \in \mathbf{MAX}(S, \succsim')$. As we also have $z\ \mathbf{R}_C\ \mathbf{MAX}(S, \succsim')$, Lemma 3.1 implies $z \in \bigcirc(\mathbf{MAX}(S, \succsim'), \mathbf{R}_C) = C(S)$. Conclusion: $C(S) \supseteq \max(S, \mathbf{R}) \cap \mathbf{MAX}(S, \succsim)$. Thus, (9) holds for any $S \in \mathfrak{X}$, and we are done.

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