Solvability of Variational Inequalities on Hilbert Lattices[*]

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Abstract

This paper provides a systematic solvability analysis for (generalized) variational inequalities on separable Hilbert lattices. By contrast to a large part of the existing literature, our approach is lattice-theoretic, and is not based on topological fixed point theory. This allows us to establish the solvability of certain types of (generalized) variational inequalities without requiring the involved (set-valued) maps be hemicontinuous or monotonic. Some of our results generalize those obtained in the context of nonlinear complementarity problems in earlier work, and appear to have scope for applications. This is illustrated by means of several applications to fixed point theory, optimization and game theory.

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1 Introduction

Topological fixed point theory is commonly used in establishing the solvability of variational inequalities. To wit, if K is a nonempty, compact and convex subset of a Hilbert space X, and $F : K \to X$ is a continuous map, then we can show that there exists an x^* in K such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \text{for every } x \in K$$

– this is the classical Hartman-Stampacchia theorem – as follows: First, we recall that an x^* in K satisfies this inequality if and only if it is a fixed point of the so-called *natural map* $\Pi_K \circ (\mathrm{id}_K - F)$, where id_K is the identity map on K and $\Pi_K : X \to K$ is the metric projection operator onto K. (This well-known fact is an immediate consequence

of the variational characterization of metric projection operators.) Second, we note that Π_K is a continuous (in fact, 1-Lipschitz) map on X, and hence $\Pi_K \circ (\mathrm{id}_K - F)$ is a continuous self-map on the compact and convex subset K of X. By the Schauder fixed point theorem, therefore, it has a fixed point, and we are done.

However, when the map F is not known to be continuous, this approach has to be modified. On the whole, the literature on variational inequalities has attempted to deal with such cases by weakening the continuity requirement to some form of hemicontinuity or related conditions; see, for instance, Ricceri (1985) and Yao and Guo (1994), among others. By contrast, the alternative approach in which solvability of variational inequalities with discontinuous maps is studied by means of order-theoretic fixed point theory has not received much attention. Exceptions to these are the works of Fujimoto (1984), Chitra and Subrahmanyam (1987), and Borwein and Demspter (1989) – these papers examine an important, albeit special, form of variational inequalities, namely nonlinear complementarity problems, from the order-theoretic angle. Put succinctly, the purpose of the present paper is to extend the order-theoretic approach to the context of all (generalized) variational inequalities, and provide solvability results without postulating any hemicontinuity conditions. Some of our results will be shown to generalize those obtained in the context of complementarity problems in the aforementioned earlier work.

It may be worth illustrating the promise of the order-theoretic approach for variational inequalities at large by revisiting the proof of the Hartman-Stampacchia theorem we sketched above. To wit, in that setup, suppose X is endowed with a partial order that is compatible with its inner product structure in a way that makes X a Hilbert lattice – all technical terms pertaining to order theory and vector lattices are explained in Section 2 below – and assume that K is, in addition, a sublattice of X with respect to this partial order. Then, provided that X is separable, it is easily checked that K is a subcomplete sublattice of X, that is, sup and inf of any nonempty subset K of X belong to K. (See Corollary 2.3 for a more general observation.) Furthermore, a result due to Isac (1995) ensures that Π_K is an order-preserving operator from K into X. (This result (for completeness), and its converse (which seems new), are proved in Lemma 2.4 below.) Consequently, if, instead of continuity of F, we ask for its order-reversion, or more generally, for the order-preservation of the map $\mathrm{id}_K - F$, we see that the natural map $\Pi_K \circ (\mathrm{id}_K - F)$ is an order-preserving self-map on the complete lattice K. It thus follows from the classical Knaster-Tarski fixed point theorem that the collection of all fixed points of this map, and hence the set of all solutions to the variational inequality at hand, is not only nonempty, but it is, in fact, a complete lattice (relative to the partial order of X). This simple argument motivates providing a systematic development of the order-theoretic approach to generalized variational inequalities and examine some related applications. As we noted above, this is the principal objective of the present paper.

The content of our work can be summarized as follows. In Section 2 we review the concepts we need from vector lattice theory. In particular, we briefly discuss Hilbert lattices, completeness of a sublattice of a Hilbert lattice, and the characterization of the order-preservation of the metric projection operator from a Hilbert lattice onto a closed and convex subset of that lattice. Mainly for completeness of the exposition, we provide proofs in this section for the results that are essential for the main body of our work.

Section 3 contains our main results on the solvability of generalized variational inequalities. In particular, we show that any generalized variational inequality with a compact-valued correspondence Γ on a weakly compact and convex sublattice of a separable Hilbert lattice X has a (maximal) solution, provided that Γ satisfies some (easy-to-check) order-theoretic conditions (Theorem 3.1). We also find that this result is equivalent to the following (seemingly new) fixed point theorem: Every (upper) order-preserving and compact-valued self-correspondence on a closed, bounded and convex sublattice of a separable Hilbert lattice has a fixed point (Theorem 3.2). Two extensions of Theorem 3.1 are also considered in Section 3. First, we provide some simple order-theoretic coercivity conditions that allow relaxing the weak compactness requirement of that theorem to mere closedness (Theorem 3.3). Second, we extend Theorem 3.1 to the context of parametric generalized variational inequalities, and provide sufficient conditions that ensure the solution correspondence of such an inequality to be order-preserving.

In Section 4 we confine our attention to variational inequalities, and observe that the results of Section 3 become sharper in this context. In particular, (a special case of) the order-preservation property we used in that section becomes equivalent to the requirement that there exist a real number $\alpha > 0$ such that

$$\alpha(x - y) \succcurlyeq F(x) - F(y) \quad \text{for every } x, y \in K \text{ with } x \succcurlyeq y,$$

where $F : K \to X$ is the map of the involved variational inequality and $\succcurlyeq$ is the partial order of the Hilbert lattice under consideration. Adopting the terminology introduced in Németh (2009), we refer to any such map F as weakly order-Lipschitz. In Section 4.1, we provide several examples of such maps, and show that the weakly order-Lipschitz property is closely related to Z-maps (which are known to play an important role in the theory of complementarity problems). In Section 4.2, we establish (by a similar argument we gave in the third paragraph of this Introduction) that the set of all solutions to a variational inequality is a complete lattice (but not a sublattice) of X, provided that the domain of this inequality is a weakly compact and convex sublattice of X and the involved map is weakly order-Lipschitz (Theorem 4.2). As in Section 3, we also examine variations of this result in which weak compactness is relaxed to closedness and parametric variational inequalities are allowed (Corollaries 4.3 and 4.4).

In Section 5 we consider a number of applications of our main results on the solvability of generalized and ordinary variational inequalities. First, we apply our results in the context of complementarity problems; this clarifies the connection between the present work and that on (order-)complementarity problems. Second, we deduce a new fixed point theorem for correspondences (Proposition 5.2) and an existence theorem for the minima of differentiable maps (Proposition 5.4). These results exemplify the potential use of our approach in that the involved correspondence in the former result and the gradients of maps in the latter are not assumed to be hemicontinuous. As a further illustration of this, and by extending the results of Section 4 to product Hilbert lattices, we provide an equilibrium existence theorem for n-person strategic games with payoff functions that may be discontinuous in others' actions (Theorem 5.6).

2 Preliminaries

We begin by briefly reviewing some order-theoretic terminology that we shall utilize in the body of the paper.

Posets and Lattices. A **poset** is an ordered pair $(X, \succcurlyeq)$ where X is a nonempty set and $\succcurlyeq$ a partial order on X. Given such a poset, and any x in X, we define $x^\uparrow := \{y \in X : y \succcurlyeq x\}$ and $x^\downarrow := \{y \in X : x \succcurlyeq y\}$. In turn, for any nonempty subset S of X, we write

$$S^\uparrow := \bigcup \{x^\uparrow : x \in S\} \quad \text{and} \quad S^\downarrow := \bigcup \{x^\downarrow : x \in S\}.$$

We say that an element x of X is an $\succcurlyeq$-**upper bound** for S if $x \succcurlyeq S$, that is, $x \succcurlyeq y$ for each $y \in S$. (The notation $S \succcurlyeq x$ is similarly understood.) We say that S is $\succcurlyeq$-**bounded from above** if $x \succcurlyeq S$ for some $x \in X$, and $\succcurlyeq$-**bounded from below** if $S \succcurlyeq x$ for some $x \in X$. In turn, S is said to be $\succcurlyeq$-**bounded** if it is $\succcurlyeq$-bounded both from above and below. As usual, we say that a sequence (x_m) in X is said to be $\succcurlyeq$-bounded (from above/below) if $\{x_1, x_2, ...\}$ possesses this property.

Given any poset $(X, \succcurlyeq)$, if $x \in S$ and $y \succcurlyeq x$ does not hold for any $y \in S \backslash \{x\}$, we say that x is a $\succcurlyeq$-**maximal** element of S. If $x \in S$ and x is an $\succcurlyeq$-upper bound for S, we say that x is the $\succcurlyeq$-**maximum** in S. (The $\succcurlyeq$-**minimum** element of S is defined similarly.) A nonempty subset S of X is said to be a $\succcurlyeq$-**chain** in X if either $x \succcurlyeq y$ or $y \succcurlyeq x$ hold for each $x, y \in S$.

For any two posets $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$, we say that a map $F : X \to Y$ is **order-preserving** if

$$x \succcurlyeq_X y \quad \text{implies} \quad F(x) \succcurlyeq_Y F(y)$$

for any $x, y \in X$. In turn, if $\Gamma : X \rightrightarrows Y$ is a correspondence – by this we mean that Γ is a map from X into $2^Y \backslash \{\emptyset\}$ – we say that Γ is **upper order-preserving** if $x \succcurlyeq_X y$ implies that for every $y' \in \Gamma(y)$ there is an $x' \in \Gamma(x)$ such that $x' \succcurlyeq_Y y'$. (**Upper order-reversing** maps are defined dually.) Similarly, Γ is **lower order-preserving** if $x \succcurlyeq_X y$ implies that for every $x' \in \Gamma(x)$ there is a $y' \in \Gamma(y)$ such that $x' \succcurlyeq_Y y'$. Γ is **order-preserving** if it is both upper and lower order-preserving. If $(X, \succcurlyeq_X)$ and $(Y, \succcurlyeq_Y)$ are subposets of a given poset $(Z, \succcurlyeq)$, then we use the phrase $\succcurlyeq$-**preserving** instead of order-preserving.

Let $(X, \succcurlyeq)$ be a poset and S a nonempty subset of X. The $\succcurlyeq$-**supremum** of S is the $\succcurlyeq$-minimum of the set of all $\succcurlyeq$-upper bounds for S, and is denoted by $\bigvee_X S$. (The $\succcurlyeq$-**infimum** of S – denoted as $\bigwedge_X S$ – is defined similarly.) As is conventional, we denote $\bigvee_X \{x, y\}$ as $x \vee y$, and $\bigwedge_X \{x, y\}$ as $x \wedge y$, for any $x, y \in X$, throughout this paper. If $x \vee y$ and $x \wedge y$ exist for every x and y in X, we say that $(X, \succcurlyeq)$ is a **lattice**, and if $\bigvee_X S$ and $\bigwedge_X S$ exist for every nonempty ($\succcurlyeq$-bounded) $S \subseteq X$, we say that $(X, \succcurlyeq)$ is a (**Dedekind**) **complete lattice**. Finally, if Y is a nonempty subset of X which contains $\bigvee_X \{x, y\}$ and $\bigwedge_X \{x, y\}$ for every $x, y \in Y$, then it is said to be a $\succcurlyeq$-**sublattice** of X. In turn, if Y contains $\bigvee_X S$ and $\bigwedge_X S$ for every nonempty $S \subseteq Y$, then it is said to be a **subcomplete** $\succcurlyeq$-**sublattice** of X. (Easy examples show that a $\succcurlyeq$-sublattice of X which happens to be a complete lattice with respect to $\succcurlyeq$ need not be a subcomplete $\succcurlyeq$-sublattice of X.)

Riesz Spaces. An **ordered linear space** is a poset $(X, \succcurlyeq)$ where X is a (real) linear space whose linear structure is compatible with the partial order $\succcurlyeq$ in the sense that $\alpha \mathrm{id}_X + z$ is a $\succcurlyeq$-preserving self-map on X for every $z \in X$ and real number $\alpha > 0$. If $(X, \succcurlyeq)$ is, in addition, a lattice, we say that $(X, \succcurlyeq)$ is a **Riesz space.** In turn, a Riesz space $(X, \succcurlyeq)$ is called a **normed Riesz space** if X is a normed linear space whose norm $\|\cdot\|$ is compatible with the partial order $\succcurlyeq$ in the sense that

$$|x| \succcurlyeq |y| \quad \text{implies} \quad \|x\| \geq \|y\|$$

for every $x, y \in X$.[1] If $(X, \succcurlyeq)$ is a normed Riesz space, it is readily checked that the lattice operations $\vee$ and $\wedge$ are continuous maps from $X \times X$ into X. As an immediate consequence of this fact, we find that the positive cone $X_+ := \{x \in X : x \succcurlyeq \mathbf{0}\}$ of $(X, \succcurlyeq)$ is a closed cone in X. (As X_+ is convex, it is weakly closed as well.) In turn, this implies that $x_m \to \bigvee_X \{x_1, x_2, ...\}$ for every $\succcurlyeq$-increasing and convergent sequence (x_m) in X. Finally, we recall that, for any convex cone C in X, the **dual cone** of C is defined as

$$C^* := \{f \in X^* : f(x) \geq 0 \text{ for every } x \in C\},$$

where X^* is the topological dual of X.

Hilbert Lattices. A normed Riesz space is called a **Banach lattice** if its norm renders the space complete. If this norm is induced by an inner product $\langle \cdot, \cdot \rangle$ on X, that is, X is a Hilbert space, we refer to $(X, \succcurlyeq)$ as a **Hilbert lattice.**

Let $(X, \succcurlyeq)$ be a Banach lattice. We say that two elements x and y of X are $\succcurlyeq$-**disjoint** if $|x| \wedge |y| = \mathbf{0}$. In turn, for any $1 \leq p < \infty$, the norm $\|\cdot\|$ of X is called p-**additive** if $\|x + y\|^p = \|x\|^p + \|y\|^p$ for any $\succcurlyeq$-disjoint $x, y \in X$. It is known that p-additivity of the norm of a Banach lattice ensures its order-continuity. In particular, if $\|\cdot\|$ is p-additive, then $(X, \succcurlyeq)$ is Dedekind complete and every $\succcurlyeq$-increasing sequence in X which is $\succcurlyeq$-bounded from above converges. (See, for instance, Meyer-Nieberg (1991), Theorem 2.4.2 and Corollary 2.4.13.) As it is easily verified that the norm of any Hilbert lattice is 2-additive, therefore, every Hilbert lattice is Dedekind complete. Furthermore, in a Hilbert lattice, an increasing (decreasing) sequence converges iff it is bounded from above (below).[2] In particular, if $(X, \succcurlyeq)$ is a Hilbert lattice and (x_m) is an $\succcurlyeq$-increasing sequence in X such that $\{x_1, x_2, ...\}$ is $\succcurlyeq$-bounded from above, then $x_m \to \bigvee_X \{x_1, x_2, ...\}$.

Let $(X, \succcurlyeq)$ be a Hilbert lattice. Then, by the Riesz representation theorem, we have $C^* = \{y \in X : \langle x, y \rangle \geq 0 \text{ for every } x \in C\}$ for any convex cone C in X. It follows easily from this observation that X_+ is self-dual in any Hilbert lattice $(X, \succcurlyeq)$, that is, $X_+ = X_+^*$. (In fact, as shown by Borwein and Yost (1984), an ordered Riesz space which happens to be a Hilbert space is a Hilbert lattice iff $X_+ = X_+^*$.) Put explicitly, we have $x \succcurlyeq \mathbf{0}$ iff $\langle x, y \rangle \geq 0$ for every $y \in X_+$. The next lemma collects some other basic (and well-known) properties of Hilbert lattices which we shall utilize routinely below.

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[1] As usual, where $\mathbf{0}$ denotes the origin of X, we define $x^+ := x \vee \mathbf{0}$, $x^- := (-x) \vee \mathbf{0}$ and $|x| := x^+ + x^-$, for any x in X.

[2] One way of seeing this is to note that every Hilbert lattice is lattice isomorphic to $L^2(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ) – see, for instance, Corollary 2.7.5 of Meyer-Nieberg (1991).

Lemma 2.1. Let $(X, \succcurlyeq)$ be a Hilbert lattice. Then, for every $x, y \in X_+$,

$$x \succcurlyeq y \text{ and } \langle x, y \rangle \leq 0 \quad \text{imply} \quad y = \mathbf{0} \tag{1}$$

and

$$x \wedge y = \mathbf{0} \quad \text{iff} \quad \langle x, y \rangle = 0. \tag{2}$$

Moreover,

$$\langle z - z \wedge w, z \vee w - z \rangle = 0 \quad \text{for any } z, w \in X. \tag{3}$$

Proof. Take any x and y in X_+. As $X_+ \subseteq X_+^*$, we have $\langle x, y \rangle \leq 0$ iff $\langle x, y \rangle = 0$, and hence $\langle x, y \rangle \leq 0$ implies that x and $-y$ are orthogonal. If, in addition, $x \succcurlyeq y$, then $x \succcurlyeq x - y \succcurlyeq \mathbf{0}$, and hence $\|x\| \geq \|x - y\|$, so the Pythagorean theorem yields

$$\|x\|^2 \geq \|x - y\|^2 = \|x\|^2 + \|y\|^2 \geq \|x\|^2,$$

which is possible only if $\|y\| = 0$, that is, $y = \mathbf{0}$. This proves (1).

Next, we recall that $|x - y| = x \vee y - x \wedge y$ (in any Riesz space). Thus, $x \wedge y = \mathbf{0}$ implies $|x - y| = x \vee y + x \wedge y = x + y$. As $x + y \in X_+$, therefore, $x \wedge y = \mathbf{0}$ entails $\|x - y\| = \|x + y\|$ which yields $\langle x, y \rangle = 0$ by the Polarization Identity. Conversely, assume that $\langle x, y \rangle = 0$. Then, as $X_+ \subseteq X_+^*$, we have $\langle x, x \wedge y \rangle \leq \langle x, y \rangle = 0$. As $x \succcurlyeq x \wedge y \succcurlyeq \mathbf{0}$, therefore, we can use what we proved in the first paragraph to conclude that $x \wedge y = \mathbf{0}$. This proves (2).

Finally, for any z and w in X, we have $z \vee w - z = w - z \wedge w$, while

$$(z - z \wedge w) \wedge (w - z \wedge w) = z \wedge w - z \wedge w = \mathbf{0},$$

so setting $x := z - z \wedge w$ and $y := w - z \wedge w$ in (2) establishes our claim.

Completeness of a Sublattice in a Hilbert Lattice. In the sequel we shall need to know when a sublattice of a Hilbert lattice is subcomplete. The following results provide some sufficient conditions for this in the case of Hilbert lattices that admit countable dense subsets.

Lemma 2.2. Let $(X, \succcurlyeq)$ be a separable Hilbert lattice and K a closed and $\succcurlyeq$-bounded $\succcurlyeq$-sublattice of X. Then, K is a subcomplete $\succcurlyeq$-sublattice of X.

Proof. Let S be a nonempty subset of K. As X is separable, so is S, that is, there is a countable dense subset T of S. We enumerate T as $\{x_1, x_2, ...\}$, and define $y_m := \bigvee_X \{x_1, ..., x_m\}$ for each positive integer m. As K is a $\succcurlyeq$-sublattice of X, it contains each y_m. It follows that (y_m) is an $\succcurlyeq$-increasing sequence in K, and hence, as K is $\succcurlyeq$-bounded from above, $y_m \to \bigvee_X T$. Clearly, as K is closed, we have $\bigvee_X T \in K$. On the other hand, for any x in S there is a self-map σ on $\mathbb{N}$ such that $x_{\sigma(m)} \to x$. Besides, $\bigvee_X T \succcurlyeq y_{\sigma(m)} \succcurlyeq x_{\sigma(m)}$, so letting $m \to \infty$ yields $\bigvee_X T \succcurlyeq x$ (because X_+ is closed). Conclusion: $\bigvee_X T \succcurlyeq S$. As $T \subseteq S$, it follows that $\bigvee_X S = \bigvee_X T \in K$. In turn, applying this finding to $-K$ yields $\bigvee_X -S \in -K$, and hence $\bigwedge_X S = -\bigvee_X -S \in K$. Conclusion: K is a subcomplete $\succcurlyeq$-sublattice of X.

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The following special case of Lemma 2.2 is particularly useful.

Corollary 2.3. Let $(X, \succcurlyeq)$ be a separable Hilbert lattice and K a weakly compact $\succcurlyeq$-sublattice of X. Then, K is a subcomplete $\succcurlyeq$-sublattice of X.

Proof. As X is separable, so is K. Let $\{x_1, x_2, ...\}$ be a countable dense set in K, define $y_m := \bigvee_X \{x_1, ..., x_m\}$ for each positive integer m, and note that (y_m) is an $\succcurlyeq$-increasing sequence in K. By the classical Eberlein-Šmulian theorem, K is weakly sequentially compact, so there is a strictly increasing self-map σ on $\mathbb{N}$ such that $(y_{\sigma(m)})$ weakly converges to some $y \in K$. Therefore, $\langle z, y_{\sigma(m)} - y \rangle \to 0$ as $m \to \infty$, while $(\langle z, y_m - y \rangle)$ is an increasing real sequence, for any z in X_+. It follows that $\langle z, y_m \rangle \to \langle z, y \rangle$ for every $z \in X_+$. As $X = X_+ - X_+$ (because $z = z^+ - z^-$ for any $z \in X$), it follows that (y_m) converges to y weakly. But then, in view of the weak closedness of X_+ and denseness of $\{x_1, x_2, ...\}$ in K, we can argue as in the proof of Lemma 2.2 to find that $y \succcurlyeq K$. As one can similarly show that K is $\succcurlyeq$-bounded from below as well, invoking Lemma 2.2 completes our proof.

We note that Lemma 2.2 is more general than Corollary 2.3 only when K is not convex. Indeed, provided that $(X, \succcurlyeq)$ is a Hilbert lattice and K is a convex $\succcurlyeq$-sublattice of X, K is closed and $\succcurlyeq$-bounded iff it is weakly compact.

Metric Projections. Let X be a Hilbert space and C a nonempty closed and convex subset C of X. A classical result of approximation theory says that there is a unique function $\Pi_C : X \to C$ which satisfies $\|x - \Pi_C(x)\| \leq \|x - y\|$ for every $y \in C$. This map is called the **metric projection operator onto** C. It is well-known that Π_C is 1-Lipschitz, and for any $x \in C$, we have the following *variational characterization* of Π_C:

$$z = \Pi_C(x) \quad \text{iff} \quad \langle z - x, z - y \rangle \leq 0 \text{ for every } y \in C. \tag{4}$$

For the order-theoretic approach to variational inequalities, it is essential to know when a metric projection operator onto a closed and convex set C in a Hilbert lattice is order-preserving. To the best of our knowledge, this question was addressed at this level of generality only by Isac (1995) who has shown that a sufficient condition for this is C be a sublattice of X. In fact, it is quite easy to show that the converse of this observation is also true. As it is essential to the development of the present paper, and because Isac (1995) is not a readily accessible paper, we next provide a complete proof of this characterization of order-preserving metric projection operators.

Lemma 2.4. Let $(X, \succcurlyeq)$ be a Hilbert lattice and C a nonempty closed and convex subset of X. Then, Π_C is $\succcurlyeq$-preserving if, and only if, C is a $\succcurlyeq$-sublattice of X.

Proof. [$\Leftarrow$] Assume that C is a $\succcurlyeq$-sublattice of X, and take any $x, y \in X$ with $x \succcurlyeq y$. We set $z := \Pi_C(x)$ and $w = \Pi_C(y)$. To derive a contradiction, suppose $z \succcurlyeq w$ is false. Then, as $z \vee w \in C$, we have $\|x - z\| < \|x - z \vee w\|$. As

$$\|x - z\|^2 = \|x - z \vee w\|^2 + \|z \vee w - z\|^2 + 2 \langle x - z \vee w, z \vee w - z \rangle,$$

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therefore, we find
$$\|z \vee w - z\|^2 < 2 \langle z \vee w - x, z \vee w - z \rangle. \tag{5}$$

On the other hand, as $z \wedge w \in C$, we have $\|y - w\| \leq \|y - z \wedge w\|$, and hence, similarly, we find
$$\|z \wedge w - w\|^2 \leq 2 \langle z \wedge w - y, z \wedge w - w \rangle.$$

Summing the two sides of this inequality with those of (5) respectively, and recalling that $z \wedge w - w = z - z \vee w$, we find that
$$
\begin{aligned}
\|z \vee w - z\|^2 \ &< \ \langle z \vee w - x, z \vee w - z \rangle + \langle z \wedge w - y, z \wedge w - w \rangle \\
&= \ \langle z \vee w - x, z \vee w - z \rangle + \langle y - z \wedge w, z \vee w - z \rangle.
\end{aligned}
$$

It follows that
$$\langle x - y - (z - z \wedge w), z \vee w - z \rangle < 0.$$

Combining this with the final part of Lemma 2.1, therefore, $\langle x - y, z \vee w - z \rangle < 0$. As both $x - y$ and $z \vee w - z$ belong to X_+, and hence X_+^*, this is a contradiction.

[$\Rightarrow$] Assume that Π_C is $\succcurlyeq$-preserving, and take any $z, w \in C$. We wish to prove that $z \vee w \in C$. Notice first that, by (4),
$$\langle \Pi_C(z \vee w) - (z \vee w), \Pi_C(z \vee w) - z \rangle \leq 0.$$

On the other hand, $\Pi_C(z \vee w) \succcurlyeq \{\Pi_C(z), \Pi_C(w)\} = \{z, w\}$ because Π_C is $\succcurlyeq$-preserving. It follows that $\Pi_C(z \vee w) \succcurlyeq z \vee w \succcurlyeq z$, and hence,
$$\Pi_C(z \vee w) - z \succcurlyeq \Pi_C(z \vee w) - z \vee w \succcurlyeq \mathbf{0}.$$

Then, setting $x := \Pi_C(z \vee w) - z$ and $y := \Pi_C(z \vee w) - (z \vee w)$ in the first part of Lemma 2.1, we find $\Pi_C(z \vee w) = z \vee w$. As C is closed, this means $z \vee w \in C$, as we sought. That $\Pi_C(z \wedge w) = z \wedge w$ is established analogously.

Generalized Variational Inequalities. Let C be a nonempty closed and convex subset of a Hilbert space X and $F : C \to X$ a function. The **variational inequality** problem associated with C and F – we denote this problem as $\mathbf{VI}(C, F)$ – is to find an $x^* \in C$ such that
$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \text{for every } x \in C. \tag{6}$$

This problem, formulated and studied first in the classical work of Hartman and Stampacchia (1966), has been investigated extensively in the literature. (For comprehensive reviews of this literature, see, for instance, Harker and Pang (1990) and Facchinei and Pang (2003) in the case where X is finite-dimensional, and Kinderlehrer and Stampacchia (1980) in the general case.)

There are various useful generalizations of the classical variational inequality problem. A particularly important one is the one in which the map under consideration is allowed to be set-valued. Put precisely, where X and C are as above, and $\Gamma : C \rightrightarrows X$ being any correspondence, the **generalized variational inequality** problem associated with C and Γ is to find an $x^* \in C$ such that there exists a $y^* \in \Gamma(x^*)$ with
$$\langle y^*, x - x^* \rangle \geq 0 \quad \text{for every } x \in C. \tag{7}$$

We refer to this problem, which was introduced to the literature by Fang and Peterson (1982), succinctly as $\mathbf{GVI}(C, \Gamma)$. In turn, any $x^* \in C$ such that (7) holds for some $y^* \in \Gamma(x^*)$ is called a **solution** to $\mathbf{GVI}(C, \Gamma)$. If there is at least one solution to it, we say that $\mathbf{GVI}(C, \Gamma)$ is **solvable**.

One of the classical methods of studying a generalized variational inequality is to transform it to a suitable fixed point problem. The following (well-known) result provides a standard way of achieving this. (A more general formulation of this lemma is proved in Section 4; see Lemma 4.6.)

Lemma 2.5. Let X be a Hilbert space and $\lambda : X \to \mathbb{R}_{++}$ any function. Let C be a nonempty closed and convex subset of X, and $\Gamma : C \rightrightarrows X$ any correspondence. Then, x^* is a solution to $\mathbf{GVI}(C, \Gamma)$ if, and only if,

$$x^* \in \ \mathrm{Fix}(\Pi_C \circ (\mathrm{id}_C - \lambda\Gamma)).$$

Many classical results on the solvability of a $\mathbf{GVI}(C, \Gamma)$ are obtained through finding conditions on the primitives C and Γ that would allow an appropriate fixed point theorem to apply to the so-called *natural correspondence* $\Pi_C \circ (\mathrm{id}_C - \Gamma)$, and then applying Lemma 2.5. We shall adopt this strategy here as well, but in contrast to the major part of the related literature, we shall posit that the underlying space has an order structure in the form of a Hilbert lattice, and use the order-theoretic fixed point theory to this end, as opposed to metric or topological fixed point theory.

3   Generalized Variational Inequalities

The Case of Bounded Domain.   Under the hypothesis that its domain is weakly compact, there are quite general order-theoretic conditions under which a given generalized variational inequality on a Hilbert lattice is solvable. The following is our main result in this regard.

Theorem 3.1. Let $(X, \succcurlyeq)$ be a separable Hilbert lattice and K a weakly compact and convex $\succcurlyeq$-sublattice of X. Let $\Gamma : K \rightrightarrows X$ be a compact-valued correspondence such that $\mathrm{id}_K - \lambda\Gamma$ is upper $\succcurlyeq$-preserving for some function $\lambda : X \to \mathbb{R}_{++}$. Then, there is a $\succcurlyeq$-maximal solution to $\mathbf{GVI}(K, \Gamma)$.

Proof. We define the correspondences $\Psi : K \rightrightarrows X$ and $f : K \rightrightarrows K$ by

$$\Psi := \mathrm{id}_K - \lambda\Gamma \quad \text{and} \quad f := \Pi_K \circ \Psi. \tag{8}$$

Let us first show that f is an upper $\succcurlyeq$-preserving correspondence. Take any x and y in K such that $x \succcurlyeq y$, and fix an arbitrary a in $f(y)$. Then $a = \Pi_K(y')$ for some $y' \in \Psi(y)$. As Ψ is upper $\succcurlyeq$-preserving by hypothesis, there is an $x' \in \Psi(x)$ with $x' \succcurlyeq y'$. But as K is a $\succcurlyeq$-sublattice of X, the map Π_K is $\succcurlyeq$-preserving by Lemma 2.4. Setting $b := \Pi_K(x')$, therefore, we find that $f(x) \ni b \succcurlyeq a$. Conclusion: f is upper $\succcurlyeq$-preserving.

We next consider the following set:

$$Y := \{x \in K : \omega \succcurlyeq x \text{ for some } \omega \in f(x)\}. \tag{9}$$

By Corollary 2.3, K is a subcomplete $\succeq$-sublattice of X. In particular, $\bigwedge_X K \in K$. It follows that $\bigwedge_X K \in Y$, that is, Y is nonempty. Consequently, $(Y, \succeq)$ is a poset. (Here $\succeq$ stands for the restriction of the original partial order $\succeq$ on X to Y, that is, we abuse notation by denoting $\succeq \cap (Y \times Y)$ also as $\succeq$.)

We wish to show that the $(Y, \succeq)$ is inductive, that is, every $\succeq$-chain in Y has an $\succeq$-upper bound in Y. To this end, take any $\succeq$-chain S in Y. Then, for any $x \in S$ there is an $\omega(x) \in f(x)$ such that $\omega(x) \succeq x$. But as $\bigvee_X S \succeq x$ for every $x \in S$, and f is upper $\succeq$-preserving, for each $x \in S$ there is a $\mu(x) \in f(\bigvee_X S)$ such that $\mu(x) \succeq \omega(x) \succeq x$. Thus:

$$S \subseteq f(\textstyle\bigvee_X S)^{\downarrow}.$$

We next claim that $\mathcal{S} := \{x^{\uparrow} \cap f(\bigvee_X S) : x \in S\}$ has the finite intersection property. Indeed, if T is a nonempty finite subset of S, then, as S, and hence T, is a $\succeq$-chain, there is an $\bar{x} \in T$ such that $\bar{x} \succeq T$. Since $\bar{x} \in f(\bigvee_X S)^{\downarrow}$, we have $y \succeq \bar{x}$ for some $y \in f(\bigvee_X S)$. By transitivity of $\succeq$, then, y belongs to $\bigcap\{x^{\uparrow} \cap f(\bigvee_X S) : x \in T\}$. It follows that $\mathcal{S}$ has the finite intersection property. But as the positive cone of a Hilbert lattice is closed, $x^{\uparrow}$ is closed in X, which means that $\mathcal{S}$ is a collection of closed subsets of $f(\bigvee_X S)$. Moreover, as Γ is compact-valued and Π_K is continuous, $f(\bigvee_X S)$ is a compact subset of K. Consequently, we may conclude that

$$\bigcap\{x^{\uparrow} \cap f(\textstyle\bigvee_X S) : x \in S\} \neq \emptyset.$$

This observation implies at once the existence of an $\succeq$-upper bound, say, ω, for S in $f(\bigvee_X S)$. Obviously, $\omega \succeq \bigvee_X S$. As $\omega \in f(\bigvee_X S)$, therefore, we find that $\bigvee_X S \in Y$. We conclude that every the $\succeq$-supremum (in X) of every $\succeq$-chain in Y belongs to Y. In particular: $(Y, \succeq)$ is inductive.

We now apply Zorn's Lemma to the poset $(Y, \succeq)$ to find a $\succeq$-maximal element x^* in Y. By definition of Y, there is a $y^* \in f(x^*)$ such that $y^* \succeq x^*$. Furthermore, as f is upper $\succeq$-preserving and $y^* \in f(x^*)$, there is a $z^* \in f(y^*)$ such that $z^* \succeq y^*$. It follows that $y^* \in Y$. As $y^* \succeq x^*$ and x^* is $\succeq$-maximal in Y, therefore, $x^* = y^*$ and hence $x^* \in f(x^*)$. Invoking Lemma 2.5, then, we find that x^* is a solution to $\mathbf{GVI}(K, \Gamma)$. As x^* is $\succeq$-maximal in Y, and Lemma 2.5 ensures that all solutions to $\mathbf{GVI}(K, \Gamma)$ are contained in Y, we may also conclude that x^* is a $\succeq$-maximal solution to $\mathbf{GVI}(K, \Gamma)$.

Remark 3.1. If, in addition to the hypotheses of Theorem 3.1, it is known that Γ is *strictly monotonic*, that is, $\langle x^* - x^{**}, y^* - y^{**} \rangle > 0$ for every distinct $x^*, x^{**} \in K$ and $(y^*, y^{**}) \in \Gamma(x^*) \times \Gamma(x^{**})$, we may conclude that there is a unique solution to $\mathbf{GVI}(K, \Gamma)$.

Remark 3.2. The hypotheses of Theorem 3.1 do not guarantee the existence of a $\succeq$-maximum solution to the involved generalized variational inequality. To illustrate, in the context of the Hilbert lattice $(\mathbb{R}^2, \geq)$, let K be the convex hull of $(0,0)$, $(1,2)$, $(2,1)$ and $(2,2)$. Clearly, K is a compact and convex $\geq$-sublattice of $\mathbb{R}^2$. On the other hand, for $\Gamma : K \rightrightarrows \mathbb{R}^2$ defined by $\Gamma(\mathbf{x}) := \{(x_1, -x_1), (-x_2, x_2)\}$, the correspondence $\mathrm{id}_K - \Gamma$ on K is upper $\geq$-preserving. Yet we have $\mathrm{Fix}(\mathrm{id}_K - \Gamma) = \{(0,0), (1,2), (2,1)\}$, which, by Lemma 2.5, is precisely the set of all solutions to $\mathbf{GVI}(K, \Gamma)$. This set does not have a $\geq$-maximum.[3]

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[3]This example was kindly communicated to us by Professor Jinlu Li.

A Fixed Point Theorem. It may be worth noting that the proof of the following fixed point theorem is implicit in that of Theorem 3.1:

Theorem 3.2. Let $(X, \succcurlyeq)$ be a separable Hilbert lattice and K a weakly compact and convex $\succcurlyeq$-sublattice of X. Then, every upper $\succcurlyeq$-preserving and compact-valued correspondence $f : K \rightrightarrows K$ has a fixed point.

As a matter of fact, Theorems 3.1 and 3.2 are equivalent. Indeed, it is plain that Theorem 3.1 follows by applying Theorem 3.2 to the correspondence f defined in (8), and then invoking Lemma 2.5. Conversely, under the hypotheses of Theorem 3.2, we can invoke Theorem 3.1 to find that $\mathbf{GVI}(K, \mathrm{id}_K - f)$ is solvable, and hence, by Lemma 2.5, there is an $x^* \in K$ such that $x^* \in \Pi_K(f(x^*)) = f(x^*)$.

Remark 3.3. An independent proof for Theorem 3.2 obtains by defining Y as in (9), verifying that $(Y, \succcurlyeq)$ is inductive (this step requires both order-preservation and compact-valuedness requirements on f), and applying Zorn's Lemma to find a $\succcurlyeq$-maximal element in Y. As shown in the last paragraph of the proof of Theorem 3.1, this element is a fixed point of f.[4]

The Case of Unbounded Domain. As every weakly compact subset of a Hilbert space is bounded, Theorem 3.1 applies only to variational inequalities whose domains are bounded. These sorts of difficulties are often handled by means of coercivity assumptions in the theory of variational inequalities. As our next result demonstrates, this can also be done in our present context, provided that we use an order-theoretic coercivity condition.

Theorem 3.3. Let $(X, \succcurlyeq)$ be a separable Hilbert lattice and C a closed and convex $\succcurlyeq$-sublattice of X. Let $\Gamma : C \rightrightarrows X$ be a compact-valued correspondence such that $\mathrm{id}_C - \lambda\Gamma$ is an upper $\succcurlyeq$-preserving for some function $\lambda : X \to \mathbb{R}_{++}$. In addition, assume that either
    (i) $\mathrm{id}_C - \lambda\Gamma$ is $\succcurlyeq$-preserving and there exist $x^\circ, x_\circ \in C$ with $x^\circ \succcurlyeq x_\circ$ and $\Gamma(x^\circ) \succcurlyeq \mathbf{0} \succcurlyeq \Gamma(x_\circ)$; or
    (ii) C has a $\succcurlyeq$-minimum and there exists an $x^\circ \in C$ with $\Gamma(x^\circ) \succcurlyeq \mathbf{0}$.
Then, $\mathbf{GVI}(C, F)$ is solvable.

Proof. We set
$$K := \{x \in C : x^\circ \succcurlyeq x \succcurlyeq x_\circ\}$$
where $x_\circ$ is as in (i) if (i) holds, and $x_\circ$ is the $\succcurlyeq$-minimum of C if (ii) holds. By Lemma 2.2, K is a subcomplete $\succcurlyeq$-sublattice of X. We next define the correspondence $f : C \rightrightarrows C$ by $f := \Pi_C \circ (\mathrm{id}_C - \lambda\Gamma)$. Again, this correspondence is upper $\succcurlyeq$-preserving in general, and $\succcurlyeq$-preserving if (i) holds.

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[4]This last step follows also from the fixed point theorem of Fujimoto (1984) which takes the inductiveness of $(Y, \succcurlyeq)$ as a hypothesis. (In Remark 3 of his paper, Fujimoto asserts that, in even a more general context than that of Theorem 3.2, this hypothesis would follow from closed-valuedness of f, but this is false even in the case of the real line. Consider, for instance, the correspondence $f : [0, 1] \rightrightarrows \mathbb{R}$ defined by $f(x) := \mathbb{R}$.)

We claim that $f(K) := \bigcup \{f(x) : x \in K\} \subseteq K$. To see this, take any x in K, and note that, as Π_C is $\succcurlyeq$-preserving (Lemma 2.4), $\Gamma(x^\circ) \succcurlyeq \mathbf{0}$ and $\lambda(x^\circ) > 0$ imply

$$x^\circ = \Pi_C(x^\circ) \succcurlyeq \Pi_C(x^\circ - \lambda(x^\circ)\Gamma(x^\circ)) = f(x^\circ).$$

Since $x^\circ \succcurlyeq x$ and f is upper $\succcurlyeq$-preserving, for any y in $f(x)$ there must exist a $y^\circ \in f(x^\circ)$ such that $y^\circ \succcurlyeq y$. As $x^\circ \succcurlyeq f(x^\circ)$, it follows that $x^\circ \succcurlyeq y$ for every $y \in f(x)$, that is, $x^\circ \succcurlyeq f(x)$. We can similarly show that $f(x) \succcurlyeq x_\circ$ in the case of (i), while $f(x) \succcurlyeq x_\circ$ holds obviously in the case of (ii). Conclusion: $f(K) \subseteq K$.

Now, by Theorem 3.1, there is a solution, say x^*, to $\mathbf{GVI}(K, \Gamma|_K)$. By Lemma 2.5, then, $x^* = \Pi_K(x^* - \lambda(x^*)y^*)$ for some $y^* \in \Gamma(x^*)$. In view of the variational characterization of Π_K, then,

$$\langle x^* - u, x^* - y \rangle \leq 0 \quad \text{for every } y \in K$$

where $u := x^* - \lambda(x^*)y^*$. As $x := \Pi_C(u) \in f(K) \subseteq K$, we thus find

$$\langle x^* - u, x^* - x \rangle \leq 0.$$

On the other hand, by definition of x and the variational characterization of Π_C, we have

$$\langle x - u, x - x^* \rangle \leq 0$$

because $x^* \in C$. Consequently,

$$\begin{aligned} \|x^* - x\|^2 &= \langle x^* - x, x^* - x \rangle \\ &= \langle x^* - u, x^* - x \rangle - \langle x - u, x^* - x \rangle \\ &\leq 0, \end{aligned}$$

that is, $x^* = x$. By definition of x, then, $x^* \in f(x^*)$, and invoking Lemma 2.5 completes the proof.

Parametric Generalized Variational Inequalities. In the context of parametric variational inequalities, a natural point of interest concerns the behavior of the set of solutions to the problem at hand as a set-valued function of the parameter. In the case where the involved map is continuous, one often inquires into the hemicontinuity properties of this solution correspondence (as in the classical Berge maximum theorem). In the order-theoretic setup, one would be concerned, instead, with the order-preservation properties of this correspondence. The following result deals with this comparative static problem in the context of generalized variational inequalities (with bounded domain) on a Hilbert lattice.

Theorem 3.4. Let $(X, \succcurlyeq_X)$ be a separable Hilbert lattice, $(\Theta, \succcurlyeq_\Theta)$ a poset, and K a weakly compact and convex $\succcurlyeq$-sublattice of X. Let $\Gamma : K \times \Theta \rightrightarrows X$ be a compact-valued correspondence that satisfy the following properties:

(i) There exists a map $\lambda : K \to \mathbb{R}_{++}$ such that $\mathrm{id}_K - \lambda\Gamma(\cdot, \theta)$ is upper $\succcurlyeq$-preserving for each $\theta \in \Theta$; and

(ii) $\Gamma(x, \cdot)$ is upper order-reversing for each $x \in K$.

Then, $\mathbf{GVI}(K, \Gamma(\cdot, \theta))$ is solvable for each $\theta \in \Theta$, and the solution correspondence $\Lambda : \Theta \rightrightarrows K$, defined by

$$\Lambda(\theta) := \{x^* \in K : x^* \text{ is a solution to } \mathbf{GVI}(K, \Gamma(\cdot, \theta))\},$$

is upper order-preserving.

Proof. Define $f : K \times \Theta \rightrightarrows X$ by

$$f(x, \theta) := \Pi_K(x - \lambda(x)\Gamma(x, \theta)),$$

where λ is the map given in hypothesis (i). Evidently, f is compact-valued, and by Lemma 2.5,

$$\Lambda(\theta) = \{x \in K : x = f(x, \theta)\} \quad \text{for every } \theta \in \Theta. \tag{10}$$

Furthermore, by Lemma 2.4 and by hypothesis (i), $f(\cdot, \theta)$ is upper $\succcurlyeq$-preserving for each $\theta \in \Theta$. In turn, $f(x, \cdot)$ is upper order-preserving for each $x \in K$. To see this, fix any $x \in K$ and $u, v \in \Theta$ with $u \succcurlyeq_\Theta v$, and pick any $x_v \in f(x, v)$. By definition of f, there exists a $y_v \in \Gamma(x, v)$ such that $x_v = \Pi_K(x - \lambda(x)y_v)$. On the other hand, since $\Gamma(x, \cdot)$ is upper order-reversing by hypothesis (ii), $y_v \succcurlyeq_X y_u$ for some $y_u \in \Gamma(x, u)$. It follows that

$$x_u := \Pi_K(x - \lambda(x)y_u) \succcurlyeq_X \Pi_K(x - \lambda(x)y_v) = x_v$$

as Π_K is $\succcurlyeq_X$-preserving (Lemma 2.4). As $x_u \in f(x, u)$, this proves that $f(x, \cdot)$ is upper order-preserving.

We now move to show that Λ is upper order-preserving. (In view of Theorem 3.1, this is enough to prove Theorem 3.4.) Take any u and v in Θ with $u \succcurlyeq_\Theta v$, and pick an arbitrary x_v in $\Lambda(v)$. We wish to find an x_u in $\Lambda(u)$ such that $x_u \succcurlyeq_X x_v$. To this end, define

$$K_v := K \cap x_v^{\uparrow}.$$

As it contains x_v, this set is nonempty. It is also readily checked that K_v is a weakly compact and convex $\succcurlyeq_X$-sublattice of X. We next claim that

$$f(x, u) \cap K_v \neq \emptyset \quad \text{for every } x \in x_v^{\uparrow}.$$

Indeed, as $x_v \in f(x_v, v)$ by (10), and $u \succcurlyeq_\Theta v$, we have $y \succcurlyeq_X x_v$ for some $y \in f(x_v, u)$ because $f(x_v, \cdot)$ is upper order-preserving. In turn, for any (arbitrarily fixed) x in $x_v^{\uparrow}$, we have $z \succcurlyeq_X y$ for some $z \in f(x, u)$ because $f(\cdot, u)$ is upper $\succcurlyeq_X$-preserving. Thus, $z \succcurlyeq_X x_v$, that is, $z \in f(x, u) \cap K_v$.

We now define the correspondence $g : K_v \rightrightarrows K_v$ by

$$g(x) := f(x, u) \cap K_v.$$

By what we have just observed, g is well-defined. As f is compact-valued and K_v is closed, g is compact-valued. It is also easily checked that g is upper $\succcurlyeq_X$-preserving. To see this, take any $x, x' \in K_v$ with $x \succcurlyeq_X x'$, and pick any y' in $g(x')$. Then, $y' \in f(x', u)$, and therefore, $y \succcurlyeq_X y'$ for some $y \in f(x, u)$ because $f(\cdot, u)$ is upper $\succcurlyeq_X$-preserving. Since $y' \in K_v$, we thus have $y \in K_v$, that is, $y \in g(x)$, as desired.

We may now apply Theorem 3.2 to g to find an $x_u \in K_v$ such that $x_u \in g(x_u) = f(x_u, u)$. But then $x_u \succcurlyeq_X x_v$ and $x_u \in \Lambda(u)$, and our proof is complete.

4 Variational Inequalities on Hilbert Lattices

Some of the results we have obtained in Section 3 become sharper in the context of variational inequalities. In particular, a special case of the order-preservation hypothesis we used in those results becomes quite simple to verify in this case. We thus begin this section by reviewing this property, and then proceed to use it for providing sufficient conditions for the solvability of variational inequalities (whose maps need be neither monotonic nor continuous.)

Weakly Order-Lipschitz Maps. As we shall demonstrate shortly, the following type of maps plays a useful role in the analysis of variational inequalities from the order-theoretic point of view.

Definition. Let $(X, \succcurlyeq)$ be an ordered linear space, C a nonempty subset of X, and $F : C \to X$ a function. We say that F is **weakly $\succcurlyeq$-Lipschitz** (or *weakly order-Lipschitz*) if there is a real number $\alpha > 0$ such that

$$\alpha(x - y) \succcurlyeq F(x) - F(y) \tag{11}$$

for every $x, y \in C$ with $x \succcurlyeq y$.

To the best of our knowledge, weakly order-Lipschitz maps have been introduced only recently by Németh (2009) to the literature on nonlinear complementarity problems. However, the importance of such maps for linear complementarity problems were noted by Borwein and Dempster (1989) who referred to them as maps of type λI. Furthermore, close relatives of such maps, namely, Z-maps have received quite bit of attention in the context of complementarity problems at large. The following definition is due to Riddell (1981).

Definition. Let $(X, \succcurlyeq)$ be a Hilbert lattice. A self-map F on X is said to be a Z-**map** if

$$(x - y) \wedge z = \mathbf{0} \quad \text{implies} \quad \langle F(x) - F(y), z \rangle \leq 0$$

for every $x, y, z \in X$.

When the Hilbert lattice under consideration is $(\mathbb{R}^n, \geq)$, it is easily seen that a linear self-map on $\mathbb{R}^n$ is a Z-map iff it is *off-diagonally antitone* in the sense of Rheinboldt (1970). On the other hand, in the context of an arbitrary Hilbert lattice $(X, \succcurlyeq)$, a linear self-map on X is a Z-map iff it satisfies *condition Z* of Cryer and Dempster (1980). Indeed, Z-maps are found of essential use for the existence/equivalence theory of complementarity problems at large; see, among others, Cryer and Demspter (1980), Riddell (1981), Borwein and Demspter (1989) and Schaible and Yao (1995).

The notion of being weakly order-Lipschitz is more universal than that of being a Z-map as the former applies to any map whose domain is an arbitrary subset of an ordered linear space while the latter applies only to self-maps on a particular type of a Riesz space. However, in the context of self-maps on a Hilbert lattice, there is a tight connection between these two notions. In particular, it is readily verified that every

weakly order-Lipschitz map is a Z-map in this context. As we show below, when the Hilbert lattice under consideration satisfies a suitable separability property, a certain converse of this also holds.

Let $(X, \succcurlyeq)$ be a Hilbert lattice. Recall that an element $a \in X_+$ is said to be an $\succcurlyeq$-**atom** of X if for every $b \in X$ with $a \succcurlyeq b \succcurlyeq \mathbf{0}$ there is a $\lambda \geq 0$ such that $b = \lambda a$. In turn, we say that $(X, \succcurlyeq)$ is **purely atomic** if the set $\mathcal{A}$ of all $\succcurlyeq$-atoms of X is nonempty and satisfies the following property:

$$x = \mathbf{0} \quad \text{whenever} \quad x \wedge a = \mathbf{0} \text{ for all } a \in \mathcal{A}.$$

(For instance, ℓ^2 is a purely atomic Hilbert lattice, but $L^2[0, 1]$ is not.)

In a separable and purely atomic Hilbert lattice, one can always find a countable complete orthonormal basis that consists of positive vectors. (In fact, the latter property is equivalent to being purely atomic for any separable Hilbert lattice.) Indeed, if $(X, \succcurlyeq)$ is a separable and purely atomic Hilbert lattice, and $\mathcal{A}$ stands for the set of all $\succcurlyeq$-atoms of X, then the set of **normalized $\succcurlyeq$-atoms** of X, that is,

$$\mathcal{U} := \{u \in \mathcal{A} : \|u\| = 1\}$$

is a countable complete orthonormal basis. Let us first show that $\mathcal{U}$ is orthonormal. Take any distinct a and b in $\mathcal{U}$, and to derive a contradiction, suppose $\langle a, b \rangle > 0$. Then, by Lemma 2.1, $a \wedge b \succ \mathbf{0}$, so, as both a and b are $\succcurlyeq$-atoms of X, there exist $\alpha, \beta > 0$ such that $a \wedge b = \alpha a$ and $a \wedge b = \beta b$. As $\|a\| = 1 = \|b\|$, these equations entail $a = b$, a contradiction. Thus: $\mathcal{U}$ is orthonormal. As every orthonormal set in a separable Hilbert lattice is countable, we conclude that $\mathcal{U}$ is a countable orthonormal set in X. It remains to prove that $\mathcal{U}$ is a complete basis. To this end, enumerate $\mathcal{U}$ as $\{u_1, u_2, ...\}$, and fix an arbitrary $x \in X$. By orthonormality, for any positive integer k,

$$\left\| \sum_{i \geq k} \langle x, u_i \rangle u_i \right\|^2 = \sum_{i \geq k} |\langle x, u_i \rangle|^2 ,$$

while the latter sum comverges to 0 as $k \to \infty$ (because, by Bessel's Inequality, the series $\sum_{i \geq 1} |\langle x, u_i \rangle|^2$ converges). It follows that $(\sum_{i=1}^m \langle x, u_i \rangle u_i)$ is a Cauchy sequence, so $y := x - \sum_{i=1}^\infty \langle x, u_i \rangle u_i \in X$. Then, $\langle y, u \rangle = \langle x, u \rangle - \langle x, u \rangle = 0$ for each $u \in \mathcal{U}$, which implies $\langle y, a \rangle = 0$ for each $a \in \mathcal{A}$. By Lemma 2.1 and pure atomicity of $(X, \succcurlyeq)$, therefore, $y = \mathbf{0}$, which means that $x = \sum_{i=1}^\infty \langle x, u_i \rangle u_i$. Thus: $\mathcal{U}$ is a countable complete orthonormal basis for X.

We next show that for any continuous self-map F on $(X, \succcurlyeq)$, weakly order-Lipschitz property is equivalent to being a Z-map, provided that the real map $x \mapsto \langle F(x), u \rangle$ is boundedly Gateaux differentiable in the direction of an normalized $\succcurlyeq$-atom of X, that is,

$$\partial_u \langle F(x), u \rangle := \lim_{\varepsilon \to 0} \varepsilon^{-1} \langle F(x + \varepsilon u) - F(x), u \rangle \tag{12}$$

exists for each $x \in X$ and $u \in \mathcal{U}$ and we have

$$\sup_{u \in \mathcal{U}} \sup_{x \in X} |\partial_u \langle F(x), u \rangle| < \infty. \tag{13}$$

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Proposition 4.1. Let $(X, \succcurlyeq)$ be a separable and purely atomic Hilbert lattice and F a continuous self-map on X. Assume that (12) holds for each $x \in X$ and $u \in \mathcal{U}$. Then, F is $\succcurlyeq$-Lipschitz if, and only if, F is a Z-map such that (13) holds.

Proof. $[\Leftarrow]$ Let F be a Z-map, and observe that $t \mapsto \langle F(x + tv), u \rangle$ is a decreasing function for any distinct $u, v \in \mathcal{U}$. (Indeed, if $s \geq t$, then, by Lemma 2.1,

$$((x + sv) - (x + tv)) \wedge u = ((s - t)v) \wedge u = \mathbf{0},$$

so, as F is a Z-map, $\langle F(x + sv), u \rangle \leq \langle F(x + tv), u \rangle$.) Now, take any $x, y \in X$ with $x \succcurlyeq y$. Using the monotonicity of the map $t \mapsto \langle F(x + tv), u \rangle$ repeatedly, we find

$$\langle F(y), u \rangle \geq \left\langle F \left(y + \sum_{v \in \mathcal{U} \setminus \{u\}} \langle x - y, v \rangle v \right), u \right\rangle, \quad u \in \mathcal{U}$$

by continuity of F. As $\mathcal{U}$ is a countable complete orthonormal basis for X, we have $x - y = \sum_{v \in \mathcal{U}} \langle x - y, v \rangle v$, whence

$$\langle F(y), u \rangle \geq \langle F (x - \langle x - y, u \rangle u), u \rangle, \quad u \in \mathcal{U}. \tag{14}$$

But, setting $\alpha := \sup_{u \in \mathcal{U}} \sup_{x \in X} |\partial_u \langle F(x), u \rangle|$ and using the mean value inequality, we have

$$\begin{aligned} \langle F(x), u \rangle - \langle F (x - \langle x - y, u \rangle u), u \rangle &\leq \alpha \langle \langle x - y, u \rangle u, u \rangle \\ &= \alpha \langle x - y, u \rangle \end{aligned}$$

so that
$$\langle F (x - \langle x - y, u \rangle u), u \rangle \geq \langle F(x), u \rangle - \alpha \langle x - y, u \rangle, \quad u \in \mathcal{U}.$$
Combining this with (14) yields

$$\langle \alpha(x - y) - (F(x) - F(y)), u \rangle \geq 0, \quad u \in \mathcal{U}.$$

As $X_+ = X_+^*$, it thus follows that $\alpha(x - y) \succcurlyeq F(x) - F(y)$. Conclusion: F is $\succcurlyeq$-Lipschitz.

$[\Rightarrow]$ Assume that F is $\succcurlyeq$-Lipschitz, and pick any $\alpha > 0$ such that (11) holds for every $x, y \in X_+$ with $x \succcurlyeq y$. Now take any $x, y, z \in X$ with $(x - y) \wedge z = \mathbf{0}$. Then, $x \succcurlyeq y$ and $z \succcurlyeq \mathbf{0}$, while $\langle x - y, z \rangle = 0$ (Lemma 2.1). In turn, the $\succcurlyeq$-Lipschitz property ensures that $\alpha(x - y) \succcurlyeq F(x) - F(y)$. Thus, as $X_+ \subseteq X_+^*$ and $z \succcurlyeq \mathbf{0}$, we have

$$0 = \langle x - y, z \rangle \geq \alpha^{-1} \langle F(x) - F(y), z \rangle.$$

Conclusion: F is a Z-map. Finally, notice that, as $\mathcal{U} \subseteq X_+$, we have $x + \varepsilon \mathcal{U} \succcurlyeq x$, so the $\succcurlyeq$-Lipschitz property implies $\alpha \varepsilon u \succcurlyeq F(x + \varepsilon u) - F(x)$, and hence $\varepsilon^{-1} \langle F(x + \varepsilon u) - F(x), u \rangle \leq \alpha$ for every $x \in X$, $u \in \mathcal{U}$ and $\varepsilon > 0$. Reasoning similarly in the case when $\varepsilon < 0$ as well, therefore, we find $|\partial_u \langle F(x), u \rangle| \leq \alpha$ for every $x \in X$ and $u \in \mathcal{U}$, that is, (13) holds.

Proposition 4.1 provides a rich source of $\succcurlyeq$-Lipschitz functions. The following two examples illustrate this point.

Example 4.1. Let $(X, \succcurlyeq)$ be a separable and purely atomic Hilbert lattice, and $F : X \to X$ a bounded linear Z-map. Then, F is $\succcurlyeq$-Lipschitz. This is an immediate consequence of Proposition 4.1. (Indeed, where $\|\cdot\|_*$ denotes the operator norm and $\mathcal{U}$ is the set of all noralized $\succcurlyeq$-atoms of X, Parseval's identity yields

$$|\langle F(u), u \rangle|^2 \leq \sum_{v \in \mathcal{U}} |\langle F(u), v \rangle|^2 = \|F(u)\|^2 \leq \|F\|_*^2$$

whence it follows from the linearity of F that $\partial_u \langle F(x), u \rangle = \langle F(u), u \rangle \leq \|F\|_*$ for any $x \in X$ and $u \in \mathcal{U}$.)

Example 4.2. Let e^i stand for the ith unit vector in $\mathbb{R}^n$, and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function such that

$$\sup_{x \in \mathbb{R}^n} |\partial_i F_i(x)| < \infty \quad \text{and} \quad \partial_j F_i(x) \leq 0$$

for each $i, j = 1, ..., n$ with $i \neq j$. (Here F_i stands for the ith component function of F, and $\partial_j F_i$ is the jth partial derivative of F_i, $i, j = 1, ..., n$.) Then, the Jacobian of F is off-diagonally antitone, that is, it is a Z-matrix. As is well-known, this is equivalent to say that F is a Z-map with respect to the usual order of $\mathbb{R}^n$. (See Proposition 8.b of Riddell (1981).) Furthermore, $\langle F(x), e^i \rangle = F_i(x)$ for each x in $\mathbb{R}^n$ and $i = 1, ..., n$. Therefore, we may apply Proposition 4.1 to conclude that F is $\geq$-Lipschitz.

In certain problems it may be easier to verify the weakly order-Lipschitz property directly. We illustrate this here in terms of the classical Nemyitskĭ operator.

Example 4.3. Let Ω be a nonempty open subset of $\mathbb{R}^n$. As usual, we make the Lebesgue space $L^2(\Omega)$ a Hilbert lattice by using the standard inner product and imposing on it the partial order $\succcurlyeq$ defined by $f \succcurlyeq g$ iff $f \geq g$ a.e. on Ω. Let C be any nonempty subset of $L^2(\Omega)$, and consider the map $F : C \to \mathbb{R}^\Omega$ defined by

$$F(f)(x) := \psi(x, f(x)), \quad x \in \Omega,$$

where $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function that satisfies the standard L^2-Carathéodory conditions:
   (a) $\psi(\cdot, t)$ is Lebesgue measurable for each $t \in \mathbb{R}$;
   (b) $\psi(x, \cdot)$ is continuous for each $x \in \Omega$;
   (c) there exist a map $a \in L^2(\Omega)$ and a number $\beta > 0$ such that $\psi(x, t) \leq a(x) + \beta |t|$ for almost all $x \in \Omega$ and all $t \in \mathbb{R}$.
   The L^2-Caratheodory conditions on ψ guarantee that $F(C) \subseteq L^2(\Omega)$. Furthermore, if we strengthen condition (b) to the following uniform Lipschitz continuity requirement, then F becomes $\succcurlyeq$-Lipschitz:

$$\alpha := \operatorname{ess\,sup}_{x \in \Omega} \operatorname{Lip}\psi(x, \cdot) < \infty, \tag{15}$$

where $\operatorname{Lip}\psi(x, \cdot)$ is the Lipschitz constant of $\psi(x, \cdot)$ if this map is Lipschitz continuous, and ∞ otherwise, To see this, take any $f, g \in C$ with $f \succcurlyeq g$. Then, by (15), there is a

(Lebesgue) null subset S of Ω, such that $f(x) \geq g(x)$ and $\mathrm{Lip}\psi(x, \cdot) \leq \alpha$ for all $x \in \Omega \backslash S$. Thus, in this case,

$$
\begin{aligned}
F(f)(x) - F(g)(x) &= \psi(x, f(x)) - \psi(x, g(x)) \\
&\leq \alpha \, |f(x) - g(x)| \\
&= \alpha(f - g)(x)
\end{aligned}
$$

for each $x \in \Omega \backslash S$, that is, $\alpha(f - g) \succcurlyeq F(f) - F(g)$, as we sought.

Variational Inequalities. The statement of Theorem 3.1 simplifies in the context of variational inequalities, that is, when Γ is single-valued. Indeed, an immediate consequence of this result is that $\mathbf{VI}(K, F)$ has a solution whenever $(X, \succcurlyeq)$ and K are as in Theorem 3.1 and $F : K \to X$ is a function such that the map $x \mapsto x - \lambda(x)F(x)$ is $\succcurlyeq$-preserving on K for some positive real map λ on K. In this case, we may also identify the order structure of the solution set for $\mathbf{VI}(K, F)$. In particular:

Theorem 4.2. Let $(X, \succcurlyeq)$ be a separable Hilbert lattice and K a weakly compact and convex $\succcurlyeq$-sublattice of X. Then, $\mathbf{VI}(K, F)$ is solvable for any $F : K \to X$ such that $\mathrm{id}_K - \lambda F$ is $\succcurlyeq$-preserving for some $\lambda : K \to \mathbb{R}_{++}$. Furthermore, for any such F, the set of all solutions to $\mathbf{VI}(K, F)$ constitutes a complete lattice relative to $\succcurlyeq$.

Proof. Let $F : K \to X$ be a function such that $\mathrm{id}_K - \lambda F$ is $\succcurlyeq$-preserving for some $\lambda : K \to \mathbb{R}_{++}$. Then, by Lemma 2.4, $f := \Pi_K \circ (\mathrm{id}_K - \lambda F)$ is a $\succcurlyeq$-preserving self-map on K. Moreover, by Corollary 2.3, K is a subcomplete $\succcurlyeq$-sublattice of X. By the classical Knaster-Tarski fixed point theorem, therefore, the set of all fixed points of f constitutes a complete lattice relative to $\succcurlyeq$. Invoking Lemma 2.5, then, concludes the proof.

Corollary 4.3. Let $(X, \succcurlyeq)$ be a separable Hilbert lattice and K a weakly compact and convex $\succcurlyeq$-sublattice of X. Then, the set of all solutions to $\mathbf{VI}(K, F)$ is a complete lattice (relative to $\succcurlyeq$) for any $\succcurlyeq$-Lipschitz $F : K \to X$.

The following example provides an immediate application of Corollary 4.3.

Example 4.4. Let $(X, \succcurlyeq)$ be a separable and purely atomic Hilbert lattice, and K a closed, bounded and convex $\succcurlyeq$-sublattice of X. Then, combining Example 4.1 and Corollary 4.3 shows that the set of all solutions to $\mathbf{VI}(K, F|_K)$ is a complete lattice relative to $\succcurlyeq$ for any bounded linear Z-map $F : X \to X$.

For variational inequalities with unbounded domains, we have the following result which is a special case of Theorem 3.2:

Corollary 4.4. Let $(X, \succcurlyeq)$ be a separable Hilbert lattice and C a closed and convex $\succcurlyeq$-sublattice of X. Let $F : C \to X$ be any $\succcurlyeq$-Lipschitz function such that either
   (i) There exist $x^\circ, x_\circ \in C$ with $x^\circ \succcurlyeq x_\circ$ and $F(x^\circ) \succcurlyeq \mathbf{0} \succcurlyeq F(x_\circ)$; or
   (ii) C has a $\succcurlyeq$-minimum and there exists an $x^\circ \in C$ with $F(x^\circ) \succcurlyeq \mathbf{0}$.
Then, $\mathbf{VI}(C, F)$ is solvable.

Example 4.5. Let $C_{a,b} := \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}$, where $-\infty \leq a_i \leq b_i \leq \infty$, $i = 1, ..., n$. (In this case, for any $F : C_{a,b} \to \mathbb{R}^n$, the problem $\mathbf{VI}(C, F)$ is said to be a *box-constrained variational inequality*; cf. Facchinei and Pang (2003).) An immediate application of Corollary 4.4 shows that if $a_i > -\infty$ for each i, then $\mathbf{VI}(C_{a,b}, F)$ is solvable for any $\succcurlyeq$-Lipschitz $F : C_{a,b} \to X$ with $F(C_{a,b}) \cap \mathbb{R}^n_+ \neq \emptyset$.

Remark 4.1. In the context of separable Hilbert lattices, Theorem 4.2 and Corollary 4.4 can jointly be seen as an extension of the earlier order-theoretic approaches to the nonlinear complementarity problem in the literature, such as Fujimoto (1984), Chitra and Subrahmanyam (1987) and Borwein and Demspter (1989). In particular, these works assume that the map $x \mapsto x - F(x)$ is order-preserving and $F(X_+) \cap X_+ \neq \emptyset$, and then go on to apply the Knaster-Tarski fixed point theorem to find a fixed point of the map $x \mapsto (x - F(x)) \vee \mathbf{0}$, which is then shown to correspond to a solution to $\mathbf{VI}(X_+, F)$. Part (ii) of Corollary 4.4 correspond precisely to this situation in the special case where $C = X_+$. In fact, our method of proof is a generalization of the method just mentioned as well. Indeed, if we set $C = X_+$ in the context of Corollary 4.4, we find $\Pi_C(x) = x \vee \mathbf{0}$ for any $x \in X$. Thus, if we assume that $x \mapsto x - F(x)$ is order-preserving, which is the same thing as saying that (11) holds with $\alpha = 1$, then the map f considered in the proof of Theorem 4.2 becomes none other than $x \mapsto (x - F(x)) \vee \mathbf{0}$.

Finally, we establish a slightly sharper version of Theorem 3.4 for variational inequalities.

Theorem 4.5. Let $(X, \succcurlyeq_X)$ be a separable Hilbert lattice, $(\Theta, \succcurlyeq_\Theta)$ a poset, and K a weakly compact and convex $\succcurlyeq$-sublattice of X. Let $F : K \times \Theta \to X$ be a function such that
   (i) there exists a real number $\lambda > 0$ such that $\mathrm{id}_K - \lambda F(\cdot, \theta)$ is $\succcurlyeq$-preserving for each $\theta \in \Theta$; and
   (ii) $F(x, \cdot)$ is order-reversing for each $x \in K$.
Then, $\mathbf{VI}(K, F(\cdot, \theta))$ is solvable for each $\theta \in \Theta$, and the solution correspondence $\Lambda : \Theta \rightrightarrows K$, defined by

$$\Lambda(\theta) := \{x^* \in K : x^* \text{ is a solution to } \mathbf{VI}(C, F(\cdot, \theta))\},$$

is order-preserving and complete lattice-valued.

Proof. In view of Theorems 3.4 and 4.2, we only need to prove that Λ is lower order-preserving. To this end, we define $f : X \times \Theta \to X$ by $f(x, \theta) := \Pi_K(x - \lambda F(x, \theta))$, and note that $\Lambda(\theta) = \{x \in K : x = f(x, \theta)\}$ for each $\theta \in \Theta$ (Lemma 2.5). Moreover, by using Lemma 2.4 and conditions (i) and (ii), we see that $f(\cdot, \theta)$ is $\succcurlyeq_X$-preserving for each $\theta \in \Theta$, and $f(x, \cdot)$ is $\succcurlyeq_\Theta$-preserving for each $x \in X$. Now, take any u and v in Θ with $u \succcurlyeq_\Theta v$, and pick an arbitrary x_u in $\Lambda(u)$. We define

$$K_u := K \cap x_u^\downarrow \quad \text{and} \quad g := f(\cdot, v)|_{K_u}.$$

(As it contains x_u, K_u is nonempty.) For any $x \in K$ with $x_u \succcurlyeq_X x$, we have

$$x_u = f(x_u, u) \succcurlyeq_X f(x, u) \succcurlyeq_X f(x, v) = g(x).$$

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Thus: $g(K_u) \subseteq K_u$. As g is $\succcurlyeq_X$-preserving, therefore, we may apply the Knaster-Tarski fixed point theorem to find an $x_v \in K_u$ with $x_v = g(x_v)$. Then $x_v \in \Lambda(v)$ and $x_u \succcurlyeq_X x_v$. Conclusion: Λ is lower order-preserving.

Variational Inequalities on Product Lattices. Given any positive integer n, the **product** of any given posets $(X_1, \succcurlyeq_1), ..., (X_n, \succcurlyeq_n)$ is defined as the poset $(\mathbf{X}, \succcurlyeq)$, where $\mathbf{X} := X_1 \times \cdots \times X_n$ and $\succcurlyeq$ is the *product* of $\succcurlyeq_1, ..., \succcurlyeq_n$, that is, $\succcurlyeq$ is the partial order on $\mathbf{X}$ defined by $\mathbf{x} \succcurlyeq \mathbf{y}$ iff $x_i \succcurlyeq_i y_i$ for each $i = 1, ..., n$. (Here, of course, $\mathbf{x} := (x_1, ..., x_n)$ and similarly for $\mathbf{y}$.) When each X_i is a Hilbert space, we make $\mathbf{X}$ a Hilbert space by using the inner product $\langle \cdot, \cdot \rangle : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^n \langle x_i, y_i \rangle_{X_i},$$

where $\langle \cdot, \cdot \rangle_{X_i}$ is the inner product of X_i for each $i = 1, ..., n$. Evidently, if each $(X_i, \succcurlyeq_i)$ is a Hilbert lattice, so is $(\mathbf{X}, \succcurlyeq)$ relative to this inner product. Finally, for any nonempty subset $\mathbf{S}$ of $\mathbf{X}$, by the **product** of the correspondences $\Gamma_i : \mathbf{S} \to X_i$, $i = 1, ..., n$, we mean the correspondence $\Gamma_1 \times \cdots \times \Gamma_n : \mathbf{S} \to \mathbf{X}$ defined by

$$(\Gamma_1 \times \cdots \times \Gamma_n)(\mathbf{x}) := \Gamma_1(\mathbf{x}) \times \cdots \times \Gamma_n(\mathbf{x}).$$

All of the results we have obtained above on (generalized) variational inequalities on Hilbert lattices extend to the context of products Hilbert lattices in a straightforward manner, but we can in fact deduce sharper results by using the special structure of such lattices. In particular, we have the following generalization of Lemma 2.5.

Lemma 4.6. Given any positive integer n, let $(\mathbf{X}, \succcurlyeq)$ be the product of the Hilbert spaces $(X_1, \succcurlyeq_1), ..., (X_n, \succcurlyeq_n)$, and $\lambda_i : \mathbf{X} \to \mathbb{R}_{++}$ any function, $i = 1, ..., n$. Let C_i be a nonempty closed and convex subset of X_i, set $\mathbf{C} := C_1 \times \cdots \times C_n$, and take any $\Gamma_i : \mathbf{C} \rightrightarrows X_i$, $i = 1, ..., n$. Then, $\mathbf{x}^*$ is a solution to $\mathbf{GVI}(\mathbf{C}, \Gamma_1 \times \cdots \times \Gamma_n)$ if, and only if,

$$\mathbf{x}^* \in \text{Fix}(\Pi_\mathbf{C} \circ (\text{id}_\mathbf{C} - (\lambda_1 \Gamma_1 \times \cdots \times \lambda_n \Gamma_n))). \tag{16}$$

Proof. Recalling the variational characterization of the metric projection operator, we see that (16) holds iff there exists a $\mathbf{y}^*$ in $\Gamma_1(\mathbf{x}^*) \times \cdots \times \Gamma_n(\mathbf{x}^*)$ such that

$$\langle \mathbf{x}^* - (\mathbf{x}^* - (\lambda_1(\mathbf{x}^*)y_1^*, ..., \lambda_n(\mathbf{x}^*)y_n^*), \mathbf{x} - \mathbf{x}^* \rangle \geq 0,$$

for each $\mathbf{x} \in \mathbf{C}$, that is,

$$\sum_{i=1}^n \lambda_i(\mathbf{x}^*) \langle y_i^*, x_i - x_i^* \rangle_{X_i} \geq 0 \quad \text{for every } \mathbf{x} \in \mathbf{C}.$$

Then, for each i, choosing $\mathbf{x}$ as the members of $\mathbf{C}$ whose jth component equals x_j^* for all $j \neq i$, we find that (16) implies

$$\langle y_i^*, x_i - x_i^* \rangle_{X_i} \geq 0 \quad \text{for every } x_i \in C_i \text{ and } i = 1, ..., n.$$

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Clearly, this entails $\langle \mathbf{y}^*, \mathbf{x} - \mathbf{x}^* \rangle \geq 0$ for every $\mathbf{x} \in \mathbf{C}$, that is, $\mathbf{x}^*$ is a solution to $\mathbf{GVI}(\mathbf{C}, \Gamma_1 \times \cdots \times \Gamma_n)$. As the converse implication follows from reversing the steps of this argument, we are done.

Replacing the role of Lemma 2.5 with Lemma 4.6 in their analyses allows us to extend all of the solvability results we have obtained above in a coordinatewise manner. Particularly useful for (game-theoretic) applications in this regard is the following generalization of Theorem 4.2:

Theorem 4.7. Given any positive integer n, let $(X_i, \succcurlyeq_i)$ be a separable Hilbert lattice, and K_i a weakly compact and convex $\succcurlyeq_i$-sublattice of X_i for each $i = 1, ..., n$. Where $\mathbf{X} := X_1 \times \cdots \times X_n$ and $\mathbf{K} := K_1 \times \cdots \times K_n$, suppose that $\mathbf{F} : \mathbf{X} \to \mathbf{K}$ is a function such that the map $\mathbf{x} \mapsto x_i - \lambda_i(\mathbf{x}) F_i(\mathbf{x})$ is order-preserving for each $i = 1, ..., n$.[5] Then, $\mathbf{VI}(\mathbf{K}, \mathbf{F})$ is solvable, and the set of all solutions to $\mathbf{VI}(\mathbf{K}, \mathbf{F})$ constitutes a complete lattice relative to the product $\succcurlyeq$ of $\succcurlyeq_1, ..., \succcurlyeq_n$.

Proof. Coordinatewise analysis readily shows that the map $\mathrm{id}_{\mathbf{K}} - (\lambda_1 F_1 \times \cdots \times \lambda_n F_n)$ is order-preserving. Thus, by Lemma 2.4, $\Pi_{\mathbf{K}} \circ (\mathrm{id}_{\mathbf{K}} - (\lambda_1 F_1 \times \cdots \times \lambda_n F_n))$ is an order-preserving map from $\mathbf{K}$ into $\mathbf{X}$ (relative to $\succcurlyeq$). As applying Corollary 2.3 coordinatewise shows that $\mathbf{K}$ is a subcomplete $\succcurlyeq$-sublattice of $\mathbf{X}$, we may thus invoke the Knaster-Tarski fixed point theorem and Lemma 4.5 to conclude the proof.

5 Applications

Complementarity Problems. Given a Hilbert lattice $(X, \succcurlyeq)$, the **nonlinear complementarity problem** associated with a map $F : X_+ \to X$ is to find an x^* such that
$$x^* \in X_+, \quad F(x^*) \in X_+ \quad \text{and} \quad \langle x^*, F(x^*) \rangle = 0.$$
This problem, which is often denoted as $\mathbf{NCP}(F)$, was introduced in the context of finite-dimensional Euclidean spaces by Cottle (1966), and has since then been investigated by numerous authors. We say that this problem is **feasible** if $x \in C$ and $F(x) \in C^*$ hold for at least one x in X.

As noted first by Karamardian (1972), $\mathbf{NCP}(F)$ is equivalent to $\mathbf{VI}(X_+, F)$. This allows us to deduce existence results for complementarity problems from the findings of Section 4. In particular, the following is an immediate consequence of Corollary 4.4.

Proposition 5.1. Given any separable Hilbert lattice $(X, \succcurlyeq)$, and $\succcurlyeq$-Lipschitz $F : X_+ \to X$, the problem $\mathbf{NCP}(F)$ is solvable if and only if it is feasible.

Remark 5.1. Proposition 5.1 is not new; it is implicit in the literature on complementarity problems. Indeed, this result is a special case of Theorem 2 of Fujimoto (1984). It also obtains upon combining Theorem 3.1 and Remark 5.1 of Chitra and Subrahmanyam (1987), and similarly, Theorem 3.4 and Remark 3.5.(c) of Borwein and Demspter (1989).

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[5]Here F_i is the ith component function of $\mathbf{F}$, $i = 1, ..., n$.

Fixed Point Theory. It is well-known that the theory of variational inequalities can be used to derive new fixed point theorems. The following result illustrates what can be achieved in this regard by means of the order-theoretic approach toward variational inequalities.

Proposition 5.2. Let $(X, \succcurlyeq)$ be a separable Hilbert lattice and K a weakly compact and convex $\succcurlyeq$-sublattice of X. Then, every upper $\succcurlyeq$-preserving correspondence $\Gamma : K \rightrightarrows X$ such that

$$|\mathrm{co}\{x, y\} \cap K| \geq 2 \quad \text{for each } x \in K \text{ and } y \in \Gamma(x) \backslash \{x\} \tag{17}$$

has a fixed point.

Proof. Let $\Gamma : K \rightrightarrows X$ be an upper $\succcurlyeq$-preserving correspondence, and set $\Phi := \mathrm{id}_K - \Gamma$. Then, $\mathrm{id}_K - \Phi$ is an upper $\succcurlyeq$-preserving correspondence from K into X. Therefore, by Theorem 3.1, we can find an $x^* \in K$ and a $y^* \in \Gamma(x^*)$ (and thus $x^* - y^* \in \Phi(x^*)$) such that

$$\langle x^* - y^*, x - x^* \rangle \geq 0 \quad \text{for every } x \in K.$$

Now assume that Γ satisfies (17), and to derive a contradiction, suppose $x^* \neq y^*$. Then, (17) implies that there is a $\lambda \in (0, 1]$ such that

$$x^* - \lambda(x^* - y^*) = (1 - \lambda)x^* + \lambda y^* \in K,$$

and hence

$$\langle x^* - y^*, x^* - \lambda(x^* - y^*) - x^* \rangle \geq 0.$$

It follows that $\lambda \langle x^* - y^*, x^* - y^* \rangle \leq 0$, that is, $\|x^* - y^*\| = 0$. But this contradicts the hypothesis that $x^* \neq y^*$. Conclusion: If F satisfies (17), we have $x^* = y^* \in \Gamma(x^*)$.

Optimization Theory. One can also use order-theoretic solvability results for variational inequalities to deduce existence theorems for certain types of optimization problems. To illustrate this, take any nonempty open and convex subset Ω of a Hilbert space X and a Fréchet differentiable map $f : \Omega \to \mathbb{R}$. We denote the Fréchet derivative of f at any given x in Ω by $\mathbf{D}_{f,x}$, and note that, as $\mathbf{D}_{f,x}$ is a continuous linear functional on X, there is a unique vector $\nabla_{f,x}$ in X such that $\mathbf{D}_{f,x}(\cdot) = \langle \nabla_{f,x}, \cdot \rangle$ by the Riesz representation theorem. Finally, we recall that a map $F : \Omega \to X$ is said to be **pseudo-monotone** if

$$\langle F(y), x - y \rangle \geq 0 \quad \text{implies} \quad \langle F(x), x - y \rangle \geq 0$$

for every $x, y \in \Omega$. Evidently, this property entails that

$$\langle F(y), x - y \rangle > 0 \quad \text{implies} \quad \langle F(x), x - y \rangle > 0$$

for every $x, y \in \Omega$.

Karamardian (1976) has shown that, where X is a finite-dimensional Euclidean space, pseudo-monotonicity of a differentiable map on X is equivalent to the pseudo-convexity of that map. (See also Karamardian and Schaible (1990).) This result extends to the case where X is an arbitrary Hilbert space by largely straightforward modifications of

Karamardian's original argument. We only present here the part of the result that we shall need below.

Lemma 5.3. Let Ω be a nonempty open and convex subset of a Hilbert space X, and $f : \Omega \to \mathbb{R}$ a Fréchet differentiable map. If the map $\omega \mapsto \nabla_{f,\omega}$ on Ω is pseudo-monotone, then f is pseudo-convex, that is,

$$\langle \nabla_{f,y}, x - y \rangle \geq 0 \quad \text{implies} \quad f(x) \geq f(y)$$

for every $x, y \in \Omega$.

Proof. Fix any x and y in Ω with $\langle \nabla_{f,y}, x - y \rangle > 0$ and $f(y) > f(x)$. By the generalized mean value theorem,[6] we have

$$f(x) - f(y) = \langle \nabla_{f,\lambda x + (1-\lambda)y}, x - y \rangle \quad \text{for some } \lambda \in (0,1).$$

Setting $z := \lambda x + (1 - \lambda)y$, therefore,

$$-\tfrac{1}{\lambda} \langle \nabla_{f,z}, y - z \rangle = \langle \nabla_{f,z}, x - y \rangle < 0,$$

so, by pseudo-monotonicity of the map $\omega \mapsto \nabla_{f,\omega}$, we have $\langle \nabla_{f,y}, y - z \rangle > 0$. But then

$$\lambda \langle \nabla_{f,y}, y - x \rangle = \langle \nabla_{f,y}, y - z \rangle > 0,$$

that is, $\langle \nabla_{f,y}, x - y \rangle < 0$, as we sought. $\qquad\blacksquare$

Combining Lemma 5.3 and our solvability theorem for variational inequalities yields the following existence theorem for minimization problems:

Proposition 5.4. Let $(X, \succcurlyeq)$ be a separable Hilbert lattice, C be a closed and convex $\succcurlyeq$-sublattice of X, and Ω an open and convex subset of X with $C \subseteq \Omega$. Let $f : \Omega \to \mathbb{R}$ be a Fréchet differentiable map such that the map $\omega \mapsto \nabla_{f,\omega}$ on Ω is pseudo-monotone and $\succcurlyeq$-Lipschitz. If there exist $x^\circ, x_\circ \in C$ with $x^\circ \geq x_\circ$ and $\nabla_{f,x^\circ} \succcurlyeq \mathbf{0} \succcurlyeq \nabla_{f,x_\circ}$, then

$$\arg\min\{f(x) : x \in C\} \neq \emptyset.$$

Proof. By Corollary 4.4, there is an $x^* \in C$ such that $\langle \nabla_{f,x^*}, x - x^* \rangle \geq 0$ for every $x \in C$. In turn, by Lemma 5.3, f is pseudo-convex, that is, $f(y) \geq f(x)$ for every $x, y \in \Omega$ with $\langle \nabla_{f,y}, x - y \rangle \geq 0$. Thus, $f(y) \geq f(x^*)$ for every $y \in C$. $\qquad\blacksquare$

We stress that the main advantage of this result is that it applies to differentiable maps whose gradients need not be continuous.

Game Theory. By a **strategic game**, we mean a list

$$[N, \{A_i, u_i\}_{i \in N}];$$

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[6] See, for instance, Ok (2007), pp. 698-9.

here N is a nonempty finite set, while for each $i \in N$, A_i is a nonempty set and u_i : $A_1 \times \cdots \times A_{|N|} \to \mathbb{R}$ is any function. As usual, we interpret N as the *set of players* while A_i and u_i stand for the *action space* and *payoff function* of player $i \in N$, respectively. In what follows, we denote the product set $A_1 \times \cdots \times A_{|N|}$ simply by $\mathbf{A}$, and designate $\mathbf{A}_{-i}$ to stand for the (ordered) product of the elements of $\{A_j : j \in N \setminus \{i\}\}$.

An element $\mathbf{x}^* := (x_1^*, ..., x_{|N|}^*)$ of $\mathbf{A}$ is said to be a **Nash equilibrium** of this game if

$$u_i(\mathbf{x}^*) \geq u_i(x_i, \mathbf{x}_{-i}^*) \quad \text{for each } x_i \in A_i \text{ and } i \in N. \tag{18}$$

(For any $i \in N$, by $(x_i, \mathbf{x}_{-i}^*)$ we mean the element of $\mathbf{A}$ which is obtained by replacing x_i^* in $\mathbf{x}^*$ with x_i.)

Let $[N, \{A_i, u_i\}_{i \in N}]$ be a strategic game such that A_i is a closed and convex subset of a Hilbert space X_i for each $i \in N$. In what follows, we shall view $\mathbf{A}$ as residing in the product of the Hilbert spaces $X_1, ..., X_{|N|}$, denoted as $\mathbf{X}$.

Fix any i in N. We say that u_i is **(continuously) Fréchet differentiable with respect to own actions** if there is an open and convex set Ω_i in X_i that contains A_i and u_i can be extended to Ω_i in such a way that $u_i(\cdot, \mathbf{x}_{-i})$ is continuously Fréchet differentiable on Ω_i for every $\mathbf{x}_{-i}$ in $\mathbf{A}_{-i}$. For any given $\mathbf{x}_{-i} \in \mathbf{A}_{-i}$, the Fréchet derivative of $u_i(\cdot, \mathbf{x}_{-i})$ at any x_i in Ω_i, that is, $\mathbf{D}_{u_i(\cdot, \mathbf{x}_{-i}), x_i}$, is identified with a unique vector $\nabla_i u_i(\mathbf{x})$ in X such that $\mathbf{D}_{u_i(\cdot, \mathbf{x}_{-i}), x_i} = \langle \nabla_i u_i(\mathbf{x}), \cdot \rangle_{X_i}$. In turn, we say that u_i is **pseudo-concave with respect to own actions** if

$$\langle \nabla_i u_i(\mathbf{x}), y_i - x_i \rangle_{X_i} \leq 0 \quad \text{implies} \quad u_i(\mathbf{x}) \geq u_i(y_i, \mathbf{x}_{-i})$$

for every $x_i, y_i \in A_i$ and $\mathbf{x}_{-i} \in \mathbf{A}_{-i}$.

The following observation is a generalization of a well-known theorem of Gabay and Moulin (1980) which characterizes the Nash equilibria of certain types of strategic games through a variational inequality. While the latter result is obtained for games with finite-dimensional action spaces and continuously differentiable payoff functions with respect to own actions, the characterization provided below allows for infinite-dimensional action spaces and payoff functions with discontinuous derivatives.

Lemma 5.5. Let $\mathcal{G} := [N, \{A_i, u_i\}_{i \in N}]$ be a strategic game such that A_i is a closed and convex subset of a Hilbert space X_i, and u_i is Fréchet differentiable and pseudo-concave with respect to own actions, for each $i \in N$. Then, where $\mathbf{F} : \mathbf{A} \to \mathbf{X}$ is defined by

$$\mathbf{F}(\mathbf{x}) := (-\nabla_1 u_1(\mathbf{x}), ..., -\nabla_{|N|} u_{|N|}(\mathbf{x})),$$

$\mathbf{x}^*$ is a Nash equilibrium of $\mathcal{G}$ if and only if it is a solution to $\mathbf{VI}(\mathbf{A}, \mathbf{F})$.

Proof. As $\mathbf{x}^*$ is a solution to $\mathbf{VI}(\mathbf{A}, \mathbf{F})$, we have

$$\sum_{i \in N} \langle \nabla_i u_i(\mathbf{x}^*), x_i - x_i^* \rangle_{X_i} \leq 0 \quad \text{for every } \mathbf{x} \in \mathbf{A}. \tag{19}$$

Choosing $\mathbf{x}$ to be $(x_i, \mathbf{x}_{-i}^*)$, then,

$$\langle \nabla_i u_i(\mathbf{x}^*), x_i - x_i^* \rangle_{X_i} \leq 0 \quad \text{for every } x_i \in A_i \text{ and } i \in N.$$

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As each u_i is pseudo-concave with respect to own actions, we thus obtain (18).

Conversely, assume that each u_i is continuously Fréchet differentiable with respect to own actions (but need not be pseudo-concave). Let $\mathbf{x}^*$ be a Nash equilibrium of $\mathcal{G}$. To derive a contradiction, assume $\langle \nabla_i u_i(\mathbf{x}^*), x_i - x_i^* \rangle_{X_i} > 0$ for some $i \in N$ and $x_i \in A_i$. Consider the map $h \in \mathbf{C}^1[0,1]$ defined by

$$h(t) := u_i(tx_i + (1-t)x_i^*, \mathbf{x}_{-i}^*).$$

As the generalized chain rule[7] implies $h'_+(0) = \langle \nabla_i u_i(\mathbf{x}^*), x_i - x_i^* \rangle_{X_i} > 0$, we find that there is a t in $(0,1)$ such that $h(t) > h(0)$. In view of the definition of h and convexity of A_i, this contradicts $\mathbf{x}^*$ being a Nash equilibrium of $\mathcal{G}$. Conclusion: $\langle \nabla_i u_i(\mathbf{x}^*), x_i - x_i^* \rangle_{X_i} \leq 0$ for all $i \in N$ and $x_i \in A_i$. Summing these inequalities over i, we obtain (19), that is, $\mathbf{x}^*$ is a solution to $\mathbf{VI}(\mathbf{A}, \mathbf{F})$.

We shall next show that combining this observation with Theorem 4.7 would yield equilibrium theorems for certain types of strategic games whose action spaces are partially ordered. To be precise, let $[N, \{A_i, u_i\}_{i \in N}]$ be a strategic game such that A_i is a closed and convex sublattice of a separable Hilbert lattice $(X_i, \succcurlyeq_i)$, and u_i is Fréchet differentiable with respect to own actions, for each $i \in N$. The following assumption is imposed on such a game.

Assumption A For every $i \in N$, there exists a map $\lambda_i : \mathbf{A} \to \mathbb{R}_{++}$ such that, for any $\mathbf{x}, \mathbf{y} \in \mathbf{A}$,

$$x_j \succcurlyeq_j y_j \text{ for each } j \in N \quad \text{imply} \quad x_i + \lambda_i(\mathbf{x})\nabla_i u_i(\mathbf{x}) \succcurlyeq_i y_i + \lambda_i(\mathbf{y})\nabla_i u_i(\mathbf{y}).$$

Our main result on the nonemptiness and structure of the set of Nash equilibria is given next.

Theorem 5.6. Let N be a nonempty finite set and $(X_i, \succcurlyeq_i)$ a separable Hilbert lattice for each $i \in N$. Let $\mathcal{G} := [N, \{A_i, u_i\}_{i \in N}]$ be a strategic game such that A_i is a closed, bounded and convex $\succcurlyeq_i$-sublattice of X_i, and u_i is Fréchet differentiable and pseudo-concave with respect to own actions, for each $i \in N$. If $\mathcal{G}$ satisfies Assumption A, then the set of Nash equilibria of this game is a complete lattice (relative to the product order on $\mathbf{X}$).

Proof. Assume that $\mathcal{G}$ satisfies Assumption A, and define the map $\mathbf{F} : \mathbf{A} \to \mathbf{X}$ as in Lemma 5.5. As this assumption is the same thing as saying that the map $\mathbf{x} \mapsto x_i - \lambda_i(\mathbf{x})F_i(\mathbf{x})$ is order-preserving for each $i \in N$, we may apply Theorem 4.7 to conclude that $\mathbf{VI}(\mathbf{K}, \mathbf{F})$ constitutes a complete lattice relative to product of the lattice orders $\succcurlyeq_1, ..., \succcurlyeq_{|N|}$. Applying Lemma 5.5, then, completes the proof.

Remark 5.2. For the existence part of Theorem 5.6, we may relax the boundedness requirement on A_is, provided that we assume the existence of two outcomes $\mathbf{x}^\circ$ and $\mathbf{x}_\circ$

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[7]See, for instance, Ok (2007), Proposition 4 and Exercise 22, pp. 687-90.

in $\mathbf{A}$ with $\mathbf{x}^\circ \succcurlyeq \mathbf{x}_\circ$ and $\nabla_i u_i(\mathbf{x}_\circ) \succcurlyeq \mathbf{0} \succcurlyeq \nabla_i u_i(\mathbf{x}^\circ)$ for each $i \in N$. (This hypothesis is suitable for modeling a strategic situation in which one's actions are desirable when their levels are sufficiently low and undesirable when their levels are sufficiently high.) That a Nash equilibrium of $\mathcal{G}$ exists under these conditions is obtained by replacing the role of Theorem 4.7 with the coordinatewise extension of Corollary 4.4 in the argument above.

Some comments on how Theorem 5.6 compares with the standard existence theorems for Nash equilibrium are in order. First, we note that most such existence theorems assume the compactness of action spaces of the players. By contrast, Theorem 5.6 assumes only weak compactness in this regard, which may be significant in applications with infinite-dimensional action spaces (as in infinitely repeated games). Let us then turn to strategic games whose action spaces are closed, bounded and convex subsets of finite-dimensional Euclidean spaces. For such games, (Rosen's generalization of) Nash's famous existence theorem says that an equilibrium is sure to exist if the payoff functions are continuous (everywhere) and quasi-concave with respect to own actions. As pseudo-concavity implies quasi-concavity – see Mangasarian (1965) – and Theorem 5.6 allows for the payoff functions depend on *others' actions* discontinuously, Nash's theorem and Theorem 5.6 are non-nested. A similar remark applies to numerous generalizations of Nash's theorem obtained in the literature on game theory, and also to the existence theorems obtained through the method of variational inequalities. To wit, Harker and Pang (1990) show that if each payoff function is differentiable and pseudo-concave with respect to own actions, and the gradients of these functions are continuous everywhere (including others' actions), then a Nash equilibrium exists. Again, Theorem 5.6 is not contained within this result because it allows the gradients of the payoff functions be discontinuous.

Finally, we recall that the sets of Nash equilibria of certain types of supermodular games are known to be complete lattices; cf. Zhou (1994) and Topkis (1998). However, it is easily checked that the payoff functions of a game that satisfies the assumptions of Theorem 5.6 need not satisfy the increasing differences property. Consequently, this result is not captured by the existence theorems obtained in the context of supermodular games either.

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