A CLASS OF DISSIMILARITY SEMIMETRICS FOR PREFERENCE RELATIONS

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Abstract. We propose a class of semimetrics for preference relations any one of which is an alternative to the classical Kemeny-Snell-Bogart metric. (We take a fairly general viewpoint about what constitutes a preference relation, allowing for any acyclic order to act as one.) These semimetrics are based solely on the implications of preferences for choice behavior, and thus appear more suitable in economic contexts and choice experiments. In our main result, we obtain a fairly simple axiomatic characterization for the class we propose. The apparently most important member of this class (at least in the case of finite alternative spaces), which we dub *the top-difference semimetric*, is characterized separately. We also obtain alternative formulae for it, and relative to this metric, compute the diameter of the space of complete preferences, as well as the best transitive extension of a given acyclic preference relation. Finally, we prove that our *preference metric spaces* cannot be isometically embedded in a Euclidean space.

1. Introduction

Being able to contrast individual preference relations on a set of choice objects is of great import for a variety of subdisciplines of economics, sociology, political science, and psychology. It is often the case that researchers wish to understand how dissimilar are the preferences of subjects that are estimated in a choice experiment, thereby getting a sense of the variability and/or polarization of preferences in the aggregate. Or, depending on the context, one may wish to have a way of determining which of two individuals is more altruistic (or resp., patient, or risk averse) by comparing their preferences to a benchmark altruistic (resp., fully patient, or risk neutral) preference relation. Similarly, we may try to understand which of two preference relations exhibits more indecisiveness among alternatives by checking how far off they are from being a complete preference relation. Or one may wish to investigate the extent to which a given preference relation violates a rationality axiom by checking how distant this relation is from the class of all preferences which satisfy that axiom.

Such considerations provide motivation for developing general methods of making dissimilarity comparisons between the family of all preference relations on a given finite set X of alternatives.[1] The most common way of doing this is by means

Date: March 08, 2022.

1991 *Mathematics Subject Classification.* Primary 06A06, 06A75; Secondary 06A07, 30L05.

Key words and phrases. metrics for preorders, Kemeny-Snell metric, weighted Kendall metrics, finite metric spaces, transitive closure.

This paper is in final form and no version of it will be submitted for publication elsewhere.

[1]We are being deliberately loose in this section about what we mean by a "preference relation" on X. Economists often take this to mean a preorder (if they wish to allow for indecisiveness), or

of equipping this family with a suitable distance function. The starting point of the related literature is the seminal work of Kemeny and Snell [20] who axiomatically proposed a distance function over linear orders on X – the order-theoretic terminology we use in this paper is outlined in Section 2.1 – which is based on counting the number of rank reversals between two such orders. (The distance between two linear orders according to this metric is twice the total number of involved rank reversals.) While its restriction to linear orders is limiting, Bogart [5] has extended this metric to the context of all partial orders on X by means of a modified system of axioms. To be precise, let us denote the indicator function of any partial order $\succsim$ on X by $I_{\succsim}$ (that is, $I_{\succsim}$ is the map on $X \times X$ with $I_{\succsim}(x,y) := 1$ if $x \succsim y$, and $I_{\succsim}(x,y) := 0$ otherwise). Then, the *Kemeny-Snell-Bogart metric* on the set of all partial orders on X is defined by

$$d_{\mathrm{KSB}}(\succsim, \succsim') = \sum_{x,y \in X} \left| I_{\succsim}(x,y) - I_{\succsim'}(x,y) \right|.$$

This metric has been found of great use in deducing a consensus ranking from a given collection of individual preferences (which may or may not leave some alternatives unranked). Moreover, the literature provides several extensions of, and alternatives to, this distance function. (See [15] for a survey of this literature.)

There are several perspectives in which two preference relations may differ from each other, and it is of course unreasonable to expect a single distance function to be sensitive to all of these. Indeed, there is an aspect, which is of utmost importance for economic analysis, that is not correctly attended by the Kemeny-Snell-Bogart metric. In economics at large, a preference relation $\succsim$ is viewed mainly as a means toward making choices in the context of various menus (i.e., nonempty subsets of the grand set X with at least two members), where a "choice" in a menu S on the basis of $\succsim$ is defined as a maximal element of S with respect to $\succsim$. Consequently, the more distinct the induced "choices" of two preference relations across menus are, there is reason to think of those preferences as being less similar. Here are two simple examples that highlight in what sense the d_{KSB} metric does not reflect this viewpoint properly.

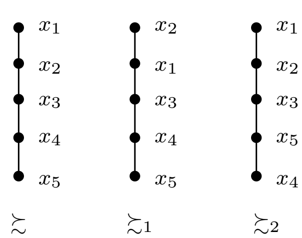

Figure 1

Example 1.1. Let $X := \{x_1, ..., x_5\}$, and consider the linear orders $\succsim$, $\succsim_1$ and $\succsim_2$ on X whose Hasse diagrams are depicted in Figure 1. Clearly, both $\succsim_1$ and $\succsim_2$

a total preorder (if they want to model the preferences of a decisive individual). By contrast, in voting theory, and operations research at large, one often assumes indifferences away, and refer to any partial or linear order as a preference relation. In this paper we work with acyclic orders, and include all of these specifications as special cases; see Section 2.2.

are obtained from $\succsim$ by reversing the ranks of two alternatives, namely, those of x_1 and x_2 in the case of $\succsim_1$ and those of x_4 and x_5 in the case of $\succsim_2$. Consequently, the Kemeny-Snell-Bogart metric judges the distance between $\succsim$ and $\succsim_1$ and that between $\succsim$ and $\succsim_2$ the same: $d_{\text{KSB}}(\succsim, \succsim_1) = 2 = d_{\text{KSB}}(\succsim, \succsim_2)$. But this conclusion is not supported from a choice-theoretic standpoint. Consider an individual whose preferences are represented by $\succsim$. This person would never choose either x_4 or x_5 in any menu $S \subseteq X$ with the exception of $S = \{x_4, x_5\}$. Consequently, the choice behavior of this person would differ from that of an individual with preferences $\succsim_2$ in only *one* menu, namely, $\{x_4, x_5\}$. By contrast, the choice behavior entailed by $\succsim$ and $\succsim_1$ are distinct in every menu that contains x_1 and x_2. So if we observed the choices made by two people with preferences $\succsim$ and $\succsim_1$, we would see them make different choices in *eight* separate menus. From the perspective of induced choice behavior, then, it is only natural that we classify "$\succsim$ and $\succsim_1$" as being less similar than "$\succsim$ and $\succsim_2$."[2,3] ‖

This example points to the fact that, at least from the perspective of choice behavior, the dissimilarity of two preferences depends not only on the number of rank reversals between them, but also *where* those reversals occur.[4]

In the next example, we illustrate that the Kemeny-Snell-Bogart metric behaves in a counterintuitive fashion (from the standpoint of induced choice behavior) also when we allow for non-comparability, or indifference, of some alternatives.

Example 1.2. Let $X := \{x_1, ..., x_4\}$, and consider the partial orders $\succsim$, $\succsim_1$ and $\succsim_2$ on X whose Hasse diagrams are depicted in Figure 2. Here $\succsim_1$ is obtained from $\succsim$ by reversing the ranks of the second-best and worst alternatives, namely, those of x_2 and x_4; we have $d_{\text{KSB}}(\succsim, \succsim_1) = 6$. On the other hand, the third preference $\succsim_2$ seems very different than $\succsim$ in that it cannot render a judgement about the relative desirability of *any* alternative; this is the preference relation of a person who is entirely indecisive about the alternatives $x_1, ..., x_4$ (whatever may be their reasons). And yet we again have $d_{\text{KSB}}(\succsim, \succsim_2) = 6$. This is, again, difficult to accept from a choice-theoretic perspective. The choices made on the bases of $\succsim$ and $\succsim_1$ differ from each other in exactly four menus. By contrast, there is no telling as to the precise nature of choices on the basis of $\succsim_2$ as every alternative in every menu is maximal with respect to this relation, so we have to declare all alternatives on a

[2]This viewpoint is also advanced in a few other papers in the literature, namely, Can [9], Hassanzadeh and Milenkovic [18], and Klamler [21]. We will clarify the connections between these papers and the present one as we proceed.

[3]As we mentioned above, there are some well-known alternatives to d_{KSB}, such as the metrics of Blin [4], Cook and Seiford [13], and Bhattacharya and Gravel [3]. These variants are also based on the idea of counting the rank reversals between two preferences in one way or another, and also yield the same conclusion as d_{KSB} in the context of this example.

[4]To put this point in a concrete perspective, recall that in the 2020 U.S. presidental elections, there were four candidates in the Electoral College: (1) D. Trump and M. Pence, (2) J. Biden and K. Harris, (3) H. Hawkins and A. N. Walker, (4) J. Jorgensen and S. Cohen. Now consider four voters each putting candidates (1) and (2) above the candidates (3) and (4). Suppose two of these voters disagree between the ranking of Trump-Pence and Biden-Harris, but agree on the relative ranking of (3) and (4), while the other two are both Trump supporters who happen to disagree on the relative ranking of (3) and (4). Obviously, in the elections, the latter two individuals both voted for the Trump-Pence ticket, while the former two casted opposite votes. And yet the Kemeny-Snell-Bogart metric views the preferences of these two pairs of voters equally distant from each other!

menu as a potential choice relative to this preference relation. But then, the choices induced by $\succsim$ and $\succsim_2$ differ at every menu.[5] $\|$

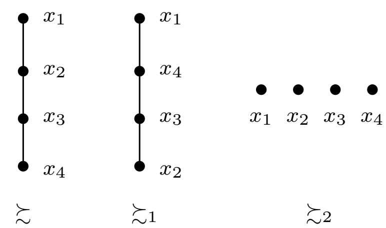

Figure 2

These examples suggest that there is room for looking at alternatives to the Kemeny-Snell-Bogart metric and its variants, especially if we wish to distinguish between preferences on the basis of their implications for choice. Our proposal here is to define a class of such alternatives looking directly at the size of the differences in choices induced by preferences across all menus, where by a "choice induced by a preference in a menu S," we mean, as usual, any maximal element in S relative to that preference. So, on a given menu S, we propose to capture the dissimilarity of two preference relations on X, say, $\succsim$ and $\trianglerighteq$, by comparing the set of $M(S,\succsim)$ of all $\succsim$-maximal elements in S with the set $M(S,\trianglerighteq)$ of all $\trianglerighteq$-maximal elements in S. A particularly simple way of making this comparison is, of course, just by counting the elements in $M(S,\succsim)$ that are not in $M(S,\trianglerighteq)$, as well as those in $M(S,\trianglerighteq)$ that are not in $M(S,\succsim)$. Thus, the number of elements in the symmetric difference $M(S,\succsim)\triangle M(S,\trianglerighteq)$ tells us how different $\succsim$ and $\trianglerighteq$ are in terms of the choice behavior they entail at the menu S. Then, summing over all menus yields the main semimetric D we propose here:

$$D(\succsim,\trianglerighteq)=\sum_{S\subseteq X}\left|M(S,\succsim)\triangle M(S,\trianglerighteq)\right|.$$

We call this map the *top-difference semimetric.*

A reinterpretion of this semimetric by using choice theory is in order. Let us first recall that a *choice correspondence* on X is any function $C:2^X\to 2^X$ with $C(S)\subseteq S$. If we abtract away from how choice correspondences come to being (via preference maximization, or boundedly rational choice procedures, or randomizations, etc.), and treat them as set-valued functions on the finite set 2^X, then the natural ℓ_1-type metric on the set of all choice correspondences on X is of the form

$$d_{\mathrm{K}}(C,C')=\sum_{S\subseteq X}\left|C(S)\triangle C'(S)\right|.$$

This metric was indeed proposed, and axiomatically characterized, by Klamler [21] (which is why we denote it by d_{K}). Now, obviously, if C and C' are rationalized

[5] A similar conclusion would hold if the third preference here declared all alternatives indifferent (instead of incomparable). In that case, the standard modification of d_{KSB} would be defined the same way but with $I_{\succsim}(x,y):=1$ if $x\succ y$ and $I_{\succsim}(x,y):=1/2$ if $x\sim y$ (where $\succ$ and $\sim$ are the asymmetric and symmetric parts of $\succsim$, respectively), and this modified metric would judge $\succsim$ and $\succsim_1$, and $\succsim$ and $\succsim_2$ (where now $x_1\sim_2 x_2\sim_2 x_3\sim_2 x_4$) equally distant, even though the choices induced by $\succsim$ and $\succsim_2$ are distinct from each other at every menu.

by preference relations $\succsim$ and $\trianglerighteq$, respectively, in the sense that $C = M(\cdot, \succsim)$ and $C' = M(\cdot, \trianglerighteq)$, then $d_{\mathrm{K}}(C, C') = D(\succsim, \trianglerighteq)$. On the other hand, it is every (nonempty-valued) choice correspondence C on X that satisfies a slight relaxation of the classical weak axiom of revealed preference is indeed of the form $S \mapsto M(S, \succsim)$ for some (transitive but possibly incomplete) preference relation $\succsim$ on X (cf. Eliaz and Ok [16]). It follows that we may think of $D(\succsim, \trianglerighteq)$ as measuring the distance between $\succsim$ and $\trianglerighteq$ by looking at the discrepancy between the "rational choices" induced by these preferences.

Having said this, counting the number of elements of $M(S, \succsim)\triangle M(S, \trianglerighteq)$ is only one way of measuring the "size" of this set. Especially if there is reason to treat the alternatives in X in a non-neutral way, we may wish to gauge this "size" by means of a measure on 2^X distinct from the counting measure.[6] This idea yields the semimetric

$$D^{\mu}(\succsim, \trianglerighteq) = \sum_{S \subseteq X} \mu(M(S, \succsim)\triangle M(S, \trianglerighteq))$$

where μ is some measure on 2^X. We refer to D^{μ} as the *μ-top-difference semimetric*. Obviously, $D^{\mu} = D$ where μ is the counting measure.

We shall show later that these semimetrics act as metrics in the case of partial orders, or complete preference relations, among other situations.[7] More important, unlike d_{KSB}, they are primed to evaluate the dissimilarity of preference relations from the perspective of choice. For instance, we have $D(\succsim, \succsim_1) = 16 > 2 = D(\succsim, \succsim_2)$ in the case of Example 1.1, while $D(\succsim, \succsim_1) = 8 < 17 = D(\succsim, \succsim_2)$ in the case of Example 1.2.[8]

One of the main advantages of the Kemeny-Snell-Bogart metric is its axiomatization. This axiomatization is not really normative, but it certainly sheds light to the basic structure of this metric by focusing on its metric segments. We begin our work in this paper by obtaining an axiomatic characterization for the class of all D^{μ} semimetrics (where μ varies over all measures on 2^X) in precisely the same spirit. Our main postulate describes exactly how the metric segments of a metric between preference relations that focus on choices may look like, and a second axiom tells us what exactly we may assess the distance between two preferences that differ from each other in the positioning of only two elements. We find that these two axioms alone characterize the entire class of D^{μ}s. (In the case one wishes to allow for indifferences, a third axiom is needed.) As in the case of the axiomatization behind d_{KSB}, the objective of these axioms is not to convince one of the appeal of a D^{μ}

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[6]Due to the political spectrum of the country, a political analyst studying voter preferences in the case of 2020 elections may wish to weigh the importance of the (1) Trump-Pence and (2) Biden-Harris tickets more than (3) Hawkins-Walker and (4) Jorgensen-Cohen tickets, *independently of voter preferences.* This analyst may then choose to use a measure μ which weighs the candidates (1) and (2) more than the candidates (3) and (4) when deciding on the size of the disagreements of the maximal sets with respect to these preferences.

[7]D^{μ} fails to distinguish between two preferences simply because indifference and incomparability sometimes have the same effect on maximal sets. For example, D^{μ} judges the difference between two preferences, one exhibiting indifference everywhere and the other incomparability everywhere, as zero. Loosely speaking, on any domain of preferences in which indifference and incomparability are not exchangeable (which is trivially the case if we assume away incomparabilities), each D^{μ} assigns a positive distance to any pair of distinct preferences.

[8]More generally, we have $D^{\mu}(\succsim, \succsim_1) > D^{\mu}(\succsim, \succsim_2)$ for every measure μ with $\mu(\{x_1, x_2\}) > \frac{1}{8}\mu(\{x_4, x_5\})$ in the context of Example 1.1, while in Example 1.2, $D^{\mu}(\succsim, \succsim_1) > D^{\mu}(\succsim, \succsim_2)$ for every measure μ.

type metric – that seems to be plain at the level of the definition of these metrics – but rather to break down what is actually involved in measuring the dissimilarity of two preferences by using D^μ. Moreover, the "if" part of this characterization is not trivial, and identifies the structure of the metric segments relative to any D^μ metric; we later use this structure in some of our applications. Finally, adding one more axiom to the system, one that reflects the neutrality of the alternatives, yields a complete characterization of the top-difference semimetric D, singling out this semimetric as a focal element of this class.

Our axioms are built on the idea of perturbing a given preference relation in a minimal way (for which the dissimilarity comparison is straightforward), and then using such perturbations finitely many times to *define* a metric segment (in terms of the target semimetric). The nature of these perturbations, and the fundamental fact that any one preference relation can be transformed into any other given preference by applying them finitely many times in the right order, is explained in Section 2.3, right after we introduce the basic nomenclature of the paper. In Section 3, we formally define our semimetrics, and show that they act as metrics in most cases of interest. And then, in Sections 3.2 and 3.3, we introduce our axiomatic system, and prove our characterization theorems. In Section 3.4, we show that D is the only member of the D^μ class which is at the same time a weighted form of the Kemeny-Snell-Bogart metric. This highlights the importance of D even further. Finally, in Section 3.5, we obtain an alternative formula for D^μ, whose computation takes at most polynomial time with respect to the size of X (just like the Kemeny-Snell-Bogart metric), and use this to obtain an efficient method of evaluating D in the case of linear orders.

When we compute the distance between preferences by D, it is difficult to understand the significance of this magnitude (or lack thereof) without a benchmark (while of course this quantity can always be used to make comparisons). For this reason, in Section 4, we turn to studying the *diameter* of certain subsets of preferences in terms of D. Even for relatively small X (with about 20 elements), there are an immense number of preference relations over X,[9] and this makes such diameter computations very hard. Fortunately, however, we were able to compute this diameter exactly (Theorem 4.1) in the case of complete preferences. When X is small (but still relevant for experimental work), the resulting diameter is quite manageable (for, say, normalization purposes).[10]

In Section 5, we turn to an application of our metrics D^μ, and study the following best approximation problem: Among all transitive extensions of an acyclic preference relation (with no indifferences), which one is the closest to that relation with respect to D^μ? We find that the answer is the transitive closure of that relation (for any μ), and provide some examples to show that this is not a trivial observation.

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[9] As a side note, we note that the number of all preorders (which is the same as that of all topologies) and the number of all partial orders (which equals that of all T_0-topologies) on an arbitrary finite set are presently known only up to sets with 16 elements. This is an intense area of research in enumerative combinatorics, but the results are mainly of asymptotic nature, as in the famous work of Kleitman and Rothschild [22].

[10] If X contains four elements, the largest D distance between two complete preferences is 26. For 5-element X this number goes up to 70, and in the 6-element case to 178. Some other computations are reported in Table 1 below.

In Section 6, we turn to the problem of isometrically embedding our preference metric spaces in a Euclidean space. When there are no indifferences, it is known that this can be done for the Kemeny-Snell-Bogart metrization. We prove in Section 6 that this is not possible in the case of D^{μ} (for any μ and any X with $|X| > 2$), so one has to adopt non-Euclidean methods when working with these metrics, such as when solving best approximation or least squares problems. The paper concludes with a short section that points to a few avenues for future research.

2. Preliminaries

2.1. **Order-Theoretic Terminology**[11]**.** By a *binary relation* R on a nonempty set X, we mean any nonempty subset of $X \times X$, but we often adopt the usual convention of writing $x\ R\ y$ instead of $(x, y) \in R$. In turn, we simply write $x\ R\ y\ R\ z$ to mean $x\ R\ y$ and $y\ R\ z$, and so on. The *principal filter* and *principal ideal* of any $x \in X$ with respect to R are defined as

$$x^{\downarrow,R} := \{a \in X : x\ R\ a\} \quad \text{and} \quad x^{\uparrow,R} := \{a \in X : a\ R\ x\},$$

respectively. When either $x\ R\ y$ or $y\ R\ x$, we say that x and y are *R-comparable*, and put

$$\text{Inc}(R) := \{(x, y) \in X \times X : x \text{ and } y \text{ are not } R\text{-comparable}\}.$$

If $\text{Inc}(R) = \varnothing$, we say that R is *total* but note that economic theorists often refer to total relations as *complete* relations.

For any $S \subseteq X$, by $x\ R\ S$, we mean $x\ R\ y$ for every $y \in S$, and interpret the statement $S\ R\ x$ analogously. The set of all *R-maximum* and *R-maximal* elements of S are denoted by $m(S, R)$ and $M(S, R)$, respectively, that is,

$$m(S, R) := \{x \in S : x\ R\ S\} \quad \text{and} \quad M(S, R) := \{x \in S : y\ R^{>}\ x \text{ for no } y \in S\},$$

where $R^{>}$ stands for the *asymmetric part* of R which is the binary relation on X defined by $x\ R^{>}\ y$ iff $x\ R\ y$ and not $y\ R\ x$. (In turn, the *symmetric part* of R is defined as $R \backslash R^{>}$.) In general, $m(S, R) \subseteq M(S, R)$, but not conversely, while $m(S, R) = M(S, R)$ whenever R is total. Note also that $M(S, R) = M(S, R^{>})$.

We denote the diagonal of $X \times X$ by Δ_X, that is,

$$\Delta_X := \{(x, x) : x \in X\}.$$

If $\Delta_X \subseteq R$, we say that R is *reflexive*, and if $R \backslash R^{>} \subseteq \Delta_X$, we say that it is *antisymmetric*. Of particular importance for the present paper is the notion of acyclicity. We say that R is *acyclic* if there do not exist any finitely many (pairwise) distinct $z_1, ..., z_k \in X$ such that $z_1\ R^{>} \cdots z_k\ R^{>}\ z_1$. This is a weaker property than transitivity. Indeed, R is said to be *transitive* if $x\ R\ y\ R\ z$ implies $x\ R\ z$, and *quasitransitive* if $R^{>}$ is transitive. It is plain that transitivity of a binary relation implies its quasitransitivity, and its quasitransitivity implies its acyclicity, but not conversely.

We say that R is a *preorder* on X if it is reflexive and transitive. (Total preorders are often called *weak orders* in the literature.) If, in addition, it is antisymmetric, R is said to be a *partial order* on X, and if it is total, antisymmetric and transitive, it is said to be a *linear order* on X. We say that R is an *acyclic order* (or sometimes an

[11]We summarize in this subsection all the order-theoretic concepts we use in this paper. However, for a comprehensive treatment of these notions we should refer the reader to authorative texts like Caspard, Leclerc and Monjardet [11] and Schröder [28].

acyclic preference) on X if it is reflexive and acyclic. In what follows, we will denote a generic acyclic order by $\succsim$ or $\trianglerighteq$, and the asymmetric parts of these relations by $\succ$ and $\vartriangleright$, respectively. We note that acyclic orders can always be identified with directed acyclic graphs, which are of primary importance for many subdisciplines of operations research.

Notation. The set of all acyclic orders on X is denoted by $\mathbb{A}(X)$, that of all preorders on X by $\mathbb{P}(X)$, and that of all total preorders by $\mathbb{P}_{\text{total}}(X)$. In turn, we denote the set of all partial orders on X by $\mathbb{P}^*(X)$, and finally, that of all linear orders on X by $\mathbb{L}(X)$. Obviously,

$$\mathbb{L}(X) \subseteq \mathbb{P}^*(X) \subseteq \mathbb{P}(X) \subseteq \mathbb{A}(X) \quad \text{and} \quad \mathbb{L}(X) \subseteq \mathbb{P}_{\text{total}}(X) \subseteq \mathbb{P}(X).$$

Finally, we recall that the *transitive closure* of a binary relation R on X is the smallest transitive relation on X that contains R; we denote this relation by $\text{tran}(R)$. This relation always exists; we have $x \ \text{tran}(R) \ y$ iff $x = x_0 \ R \ x_1 \ R \cdots R \ x_k = y$ holds for some nonnegative integer k and $x_0, ..., x_k \in X$. Obviously, $\text{tran}(R)$ is a preorder on X, provided that R is reflexive.

2.2. **Preferences.** The standard practice of economics is to model the preference relation of an individual as a total preorder. When one is interested in modeling the indecisiveness of an individual over some alternatives (as in the literature on incomplete preferences that started with Aumann [1]), or wish to model incomparability of some alternatives (because the outside observer has limited data), a preference relation is taken as any preorder on the alternative set X. There are also many studies, say, in voting theory and stable matching, where the space of preferences is identified with that of all linear orders, or partial orders.

In all these situations, the preferences are assumed to be transitive. This stems from focusing on "rational" preferences, but on closer scrutiny, one observes that transitivity is often a sufficient (and very convenient) property, but there are weaker alternatives to it. For instance, one major problem with non-transitive preferences is that these may not be maximized on some finite menus, but the following well-known, and easily proved, fact shows that this is not a cause for concern in the case of acyclic orders.

Lemma 2.1. *Let X be a nonempty set and R a reflexive binary relation on X. Then, $M(S,R) \neq \varnothing$ for every nonempty finite $S \subseteq X$ if, and only if, R is acyclic.*

Another common rationality argument for transitivity is through the so-called money pump arguments, but these too do not work against the property of acyclicity. In addition, the literature on choice theory provides plenty of rationality axioms that justify the acyclicity of *revealed* preferences; see, among many others, [29, 19]. In what follows, therefore, we model preferences on X as acyclic orders on X. This admits all of the standard ways of modeling preferences in economics as special cases, and still reflect due rationality on the part of the individuals.

As we discussed in Section 1, our primary objective is to turn $\mathbb{A}(X)$ into a (semi)metric space in a way that semimetric of the space reflect the dissimilarity of two acyclic preferences on the basis of their implications for choice. We do this in the context of a finite set of alternatives. Thus, henceforward, we always take X as a finite set that contains at least two elements, unless otherwise is explicitly stated.

(We denote the cardinality of X by n.) By a *menu* in X, we mean any $S \subseteq X$ with $|S| \geq 2$.

2.3. **Perturbations of Acyclic Preferences.** Let $\succsim$ be an acyclic order on X, and take any distinct $a, b \in X$. Suppose first that a and b are not $\succ$-comparable. In that case we define

$$R = \begin{cases} \succsim \sqcup \{(a,b)\}, & \text{if } (a,b) \in \text{ Inc}(\succsim), \\ \succsim \backslash \{(b,a)\}, & \text{if } b \sim a, \end{cases}$$

which is a binary relation on X that may or may not be acyclic.[12] (Here $\sim$ stands for the symmetric part of $\succsim$.) Provided that it is acyclic, we say that R is *obtained from* $\succsim$ *by a single addition* (of (a,b)), and denote it as

$$\succsim \oplus (a,b).$$

In words, $\succsim \oplus (a,b)$ is the acyclic order on X that is obtained from $\succsim$ by placing a strictly above b (while $\succsim$ itself does not render a strict ranking between a and b). See Figures 3 and 4.

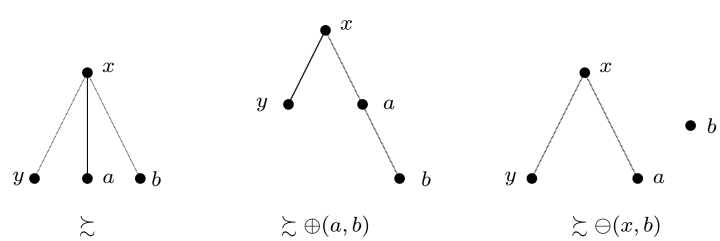

Figure 3

Now suppose $a \succ b$ holds instead. Then $\succsim \backslash \{(a,b)\}$ is acyclic, and in this case we say that this relation is *obtained from* $\succsim$ *by a single deletion* (of (a,b)), and denote it as

$$\succsim \ominus (a,b).$$

In words, $\succsim \ominus (a,b)$ is the acyclic order on X that is obtained from $\succsim$ by eliminating the strictly higher ranking of a over b within $\succsim$. See Figures 3 and 4.

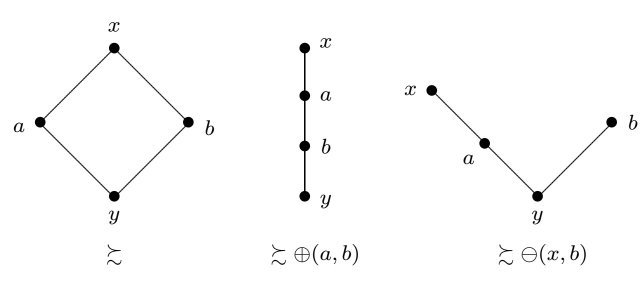

Figure 4

[12] For instance, where $X = \{a, b, c\}$, the binary relation $\succsim := \{(c,b),(b,a)\} \sqcup \triangle_X$ belongs to $\mathbb{A}(X)$, but $\succsim \oplus (a,b)$ does not.

We emphasize that both $\succsim \oplus (a,b)$ and $\succsim \ominus (a,b)$ belong to $\mathbb{A}(X)$. (In the first case this is true by definition, and in the second case this is true by necessity.) Moreover, when a and b are not $\succsim$-comparable, we have $(\succsim \oplus (a,b)) \ominus (a,b) = \succsim$, and similarly, when $a \succ b$, we have $(\succsim \ominus (a,b)) \oplus (a,b) = \succsim$. However, when $a \sim b$, we have $(\succsim \oplus (a,b)) \ominus (a,b) = \succsim \backslash \{(a,b),(b,a)\}$.

Let $\succsim_0$ and $\trianglerighteq$ be two acyclic orders on X. We say that $\succsim_0$ is a *one-step perturbation of* $\succsim$ *toward* $\trianglerighteq$ if either (i)

$$\succsim_0 \;=\; \succsim \ominus (a,b) \quad \text{and} \quad \text{not } a \vartriangleright b \tag{1}$$

and

$$x \succ b \quad \text{for every } x \in X \text{ with } x \vartriangleright b \tag{2}$$

for some $(a,b) \in \succ$; or (ii)

$$\succsim_0 \;=\; \succsim \oplus (a,b) \quad \text{and} \quad a \vartriangleright b \tag{3}$$

for some $(a,b) \in \text{Inc}(\succ)$. Intuitively speaking, when this is the case, we understand that the ranking positions of a and b in $\succsim$ is altered in a way that becomes identical to how these elements are ranked by $\trianglerighteq$. (This is captured by (1) and (3).) In this sense, we think of $\succsim_0$ as "more similar" to $\trianglerighteq$ than $\succsim$ is. This viewpoint is further enforced by the requirement (2) which maintains that the ordering of b in $\succsim$ is consistent with that in $\trianglerighteq$. The following example highlights the importance of this consistency condition.

Example 2.1. Let $X = \{a,b,c\}$, and consider the acyclic orders $\succsim$ and $\trianglerighteq$ on X with $\succsim := \Delta_X \sqcup \{(a,b)\}$ and $\trianglerighteq := \Delta_X \sqcup \{(c,b)\}$. Then, $\Delta_X = \succsim \ominus (a,b)$, but Δ_X is not a one-step perturbation of $\succsim$ toward $\trianglerighteq$. Indeed, in this case, it is not really evident whether or not Δ_X is "more similar" to $\trianglerighteq$ than $\succsim$ is, especially if we focus on the maximal elements in various subsets of X. If we restrict attention to the sets $\{a,c\}$ and $\{c,b\}$, the behavior of $\succsim$ and Δ_X are identical, while on $\{a,b\}$ the behavior of Δ_X is identical to that of $\trianglerighteq$. However, on the grand set X, the diagonal relation Δ_X behaves quite differently than $\trianglerighteq$. Indeed, Δ_X declares b as maximal in X, while b is minimal in X relative to $\trianglerighteq$. By contrast, $\succsim$ and $\trianglerighteq$ have the same set of maximal elements in X. We impose the consistency condition (2) on one-step perturbations precisely to avoid such ambiguous situations. $\|$

In what follows, if $\succsim_0$ is a one-step perturbation of $\succsim$ toward $\trianglerighteq$, we write

$$\succsim \;\rightarrow\; \succsim_0 \;\twoheadrightarrow\; \trianglerighteq .$$

Generalizing this concept, for any integer $n \geq 2$, we say an acyclic order $\succsim_{n-1}$ on X is an *n-step perturbation of* $\succsim$ *toward* $\trianglerighteq$, if there exist $\succsim_0, ..., \succsim_{n-2} \in \mathbb{A}(X)$ such that $\succsim \rightarrow \succsim_0 \twoheadrightarrow \trianglerighteq$ and

$$\succsim_{k-1} \;\rightarrow\; \succsim_k \;\twoheadrightarrow\; \trianglerighteq \;\text{ for each } k = 1, ..., n-1.$$

Finally, we say that an $\succsim_* \in \mathbb{A}(X)$ is *in-between* $\succsim$ and $\trianglerighteq$ if $\succsim_*$ is an *n-step perturbation of* $\succsim$ *toward* $\trianglerighteq$ for some positive integer n. And if $\succsim_* = \trianglerighteq$ here, we say that $\succsim$ is *transformed into* $\trianglerighteq$ *in finitely many steps.*

Remark. In the literature on metrics on preference relations, one often says that a binary relation R_0 on X is "between" the binary relations R_* and R^* on X if $R_* \cap R^* \subseteq R \subseteq R_* \cup R^*$.(See, for instance, [5, 6, 15].) Our definition of being "in-between" is more stringent than this concept, due to the consistency condition (2).

For instance, in the context of Example 2.1, Δ_X is "between" $\succsim$ and $\trianglerighteq$ according to the betweenness definition of the literature, but Δ_X is not in-between $\succsim$ and $\trianglerighteq$ according to our definition. This is consistent with the main motivation of the present work. We would like to think of an acyclic order $\succsim_*$ on X that is in-between $\succsim$ and $\trianglerighteq$ as one that is "more similar" in its order structure to $\trianglerighteq$ than $\succsim$ is. As we have seen in Example 2.1, at least insofar as which elements are declared maximal in various menus, being "between" two acyclic orders does not fully support this interpretation.

The following result provides the fundamental force behind the axiomatization that we present in the next section.

Theorem 2.2. *Let $\succsim$ and $\trianglerighteq$ be distinct acyclic orders on X with the same symmetric parts. Then, $\succsim$ can be transformed into $\trianglerighteq$ by finitely many one-step perturbations.*[13]

Proof. We will prove that there exists an $\succsim_0 \in \mathbb{A}(X)$ such that $\succsim \to \succsim_0 \twoheadrightarrow \trianglerighteq$. The more general statement of the theorem will then follow by induction.

Note first that if $\succ \subseteq \triangleright$, then the containment is proper (because $\succsim \neq \trianglerighteq$), so we are readily done by setting $\succsim_0 := \succsim \oplus (a,b)$ for any $a, b \in \triangleright \setminus \succ$. Let us then assume that $\succ$ is not contained within $\triangleright$, that is,

$$B := \{b \in X : (a,b) \in \succ \setminus \triangleright \text{ for some } a \in X\} \neq \varnothing.$$

We pick any tran($\succ$)-minimal element b^* of B, and any $a^* \in X$ with $a^* \succ b^*$ but not $a^* \triangleright b^*$. If

$$x \succ b^* \quad \text{ for every } x \in X \text{ with } x \triangleright b^*,$$

then we are done by setting $\succsim_0 := \succsim \ominus (a,b)$. We thus assume that this is not the case, that is, there is an $x \in X$ such that

$$x \triangleright b^* \quad \text{ and } \quad \text{not } x \succ b^*. \tag{4}$$

Next, we define $\succsim_0 := \succsim \sqcup \{(x, b^*)\}$. Given that $x \triangleright b^*$, our proof will be complete if we can show that $\succsim_0 = \succsim \oplus (x, b^*)$. But note that we cannot have $b^* \succ x$ here, because otherwise $x \in B$, and $b^* \succ x$ contradicts the tran($\succ$)-minimality of b^* in B. We cannot have $b^* \sim x$ either, because $x \triangleright b^*$ while $\sim$ equals to the symmetric part of $\trianglerighteq$ by hypothesis. Thus: $(x, b^*) \in \text{Inc}(\succsim)$. By definition of the relation $\succsim \oplus (x, b^*)$, it thus remains only to show that $\succsim_0$ is acyclic. To derive a contradiction, suppose this is not the case, that is, assume there exist an $n \in \mathbb{N}$ and distinct $z_1, ..., z_n \in X$ with $z_1 \succ_0 \cdots \succ_0 z_n \succ_0 z_1$. Since $\succsim \in \mathbb{A}(X)$, we must have $(z_k, z_{k+1 (\mathrm{mod}\, n)}) = (x, b^*)$ for some $k = 1, ..., n$. Thus, relabelling if necessary, we may assume that $(z_1, z_2) = (x, b^*)$ in which case we have

$$b^* = z_2 \succ \cdots \succ z_n \succ z_1 \tag{5}$$

by definition of $\succsim_0$. Now, if $z_k \triangleright z_{k+1 (\mathrm{mod}\, n)}$ for each $k = 2, ..., n$, then

$$b^* = z_2 \triangleright \cdots \triangleright z_n \triangleright z_1 = x \triangleright b^*$$

and we contradict the acyclicity of $\trianglerighteq$. Let us then assume that $z_k \triangleright z_{k+1 (\mathrm{mod}\, n)}$ fails for some $k = 2, ..., n$. In view of (5), this means that $z_k \in B$ for some $k \in$

[13]Example 2.1 points to the nontriviality of this claim. Arbitrary addition and/or deletions of pairs of alternatives from $\succsim$ may not be able to transform $\succsim$ into $\trianglerighteq$. Instead, the theorem claims that there is always a "right" order of doing these perturbations which would transform $\succsim$ into $\trianglerighteq$.

$\{1, ..., n\}\backslash\{2\}$. But again by (5), we have b^* tran($\succ$) z_k for every $k \in \{1, ..., n\}\backslash\{2\}$, so this finding contradicts the tran($\succ$)-minimality of b^* in B. We conclude that $\succsim_0$ $\in \mathbb{A}(X)$. As noted above, this completes the proof. □

In Figure 5, we provide a simple illustration of how a partial order (in this case the pentagon lattice) is transformed into another by means of three one-step perturbations. In this example, the middle two partial orders are in-between left-most and right-most partial orders. (In particular, we have $\succsim^* = ((\succsim \oplus(y, z)) \ominus (w, a)) \ominus (z, a)$.) But despite what this example may suggest, we emphasize that a non-transitive (but always acyclic) binary relation may be in-between two partial orders.

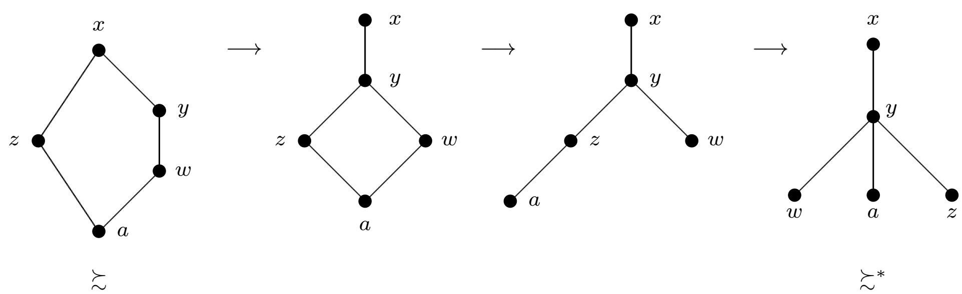

Figure 5

3. A Class of Dissimilarity Semimetrics for Preferences

3.1. **Top-Difference Semimetrics.** For any positive measure μ on 2^X,[14] we define the **μ-top-difference semimetric** $D^\mu : \mathbb{A}(X) \times \mathbb{A}(X) \to [0, \infty)$ by

$$D^\mu(\succsim, \trianglerighteq) := \sum_{S \subseteq X} \mu(M(S, \succsim) \triangle M(S, \trianglerighteq)).$$

In the special case where μ is the counting metric, we refer to D^μ simply as the **top-difference semimetric**, and denote it by D, that is,

$$D(\succsim, \trianglerighteq) := \sum_{S \subseteq X} |M(S, \succsim) \triangle M(S, \trianglerighteq)| \tag{6}$$

for any $\succsim, \trianglerighteq \in \mathbb{A}(X)$.

That each D^μ is indeed a semimetric on $\mathbb{A}(X)$ is straightforward. Unless X is a singleton, however, D^μ does not act as a metric even on $\mathbb{P}(X)$. For instance, D^μ cannot distinguish between complete indifference and complete incomparability, that is, $D^\mu(\Delta_X, X \times X) = 0$ for any measure μ on 2^X while Δ_X and $X \times X$ are distinct preorders on X when $|X| \geq 2$. (This is simply because the maximal elements relative to these relations are the same in every menu.) For another example, note that a partial order and a preorder on X may have the same asymmetric part, but may nevertheless be distinct relations on X.

[14] We consider the zero measure $S \mapsto 0$ on 2^X as a member of the family of all positive measures on 2^X, but, of course, the semimetric D^μ where μ is the zero measure is of no interest.

In passing, we note that there are interesting subclasses of acyclic orders on which D^μ acts as a metric, provided that μ has full support. We present two examples to illustrate.

Example 3.1. Any D^μ acts as a metric on the set of all partial orders on X. That is, $D^\mu|_{\mathbb{P}^*(X)\times\mathbb{P}^*(X)}$ is a metric on $\mathbb{P}^*(X)$ for any measure μ on 2^X. ‖

Example 3.2. For any preorder $\succsim$ on X, the *indifference part* of $\succsim$, denoted by ind($\succsim$), is the binary relation on X defined by $(x, y) \in \text{ind}(\succsim)$ iff

$$x \succ z \text{ iff } y \succ z \quad \text{ and } \quad z \succ x \text{ iff } z \succ y$$

for every $z \in X$. (If we interpret $\succsim$ as the preference relation of a person, then $(x, y) \in \text{ind}(\succsim)$ means that this individual treats x and y as identical objects in every menu; see [16] and [26].)

It is immediate from this definition that ind($\succsim$) is an equivalence relation on X, and $\sim \subseteq \text{ind}(\succsim)$. If $\succsim$ is total, then this holds as an equality, but in general, it may well hold properly.[15] Those preorders whose symmetric parts match their indifference parts exactly are of immediate interest for decision theory. Eliaz and Ok [16] refer to a preorder $\succsim$ on X with this property, that is, when $\sim = \text{ind}(\succsim)$, as a *regular* preorder on X.

Let $\succsim_1$ and $\succsim_2$ be two regular preorders on X such that $M(S, \succsim_1) = M(S, \succsim_2)$ for every doubleton $S \subseteq X$. We claim that $\succsim_1 = \succsim_2$. To see this, note that for any distinct $x, y \in X$, we have $x \succ_i y$ iff $\{x\} = M(\{x, y\}, \succsim_i)$ for $i = 1, 2$. By hypothesis, therefore, $\succ_1 = \succ_2$. But then, by definition of ind($\cdot$), we have $\text{ind}(\succsim_1) = \text{ind}(\succsim_2)$ as well. Since both $\succsim_1$ and $\succsim_2$ are regular, it follows that $\sim_1 = \sim_2$.

As an immediate consequence of this observation, we see that the restriction of D^μ to the class of all regular preorders on X yields a metric on that class, for any measure μ on 2^X. In particular, each D^μ is a metric on the set $\mathbb{P}_{\text{total}}(X)$ of all complete preorders, the standard setup of economic theory. ‖

3.2. **Axioms.** In this section, we discuss a few properties of metrics on $\mathbb{A}(X)$ which we will use to characterize the top-difference metrics. At the outset, however, we would like to point out that our objective here is not to "justify" these metrics; we do not necessarily see the following axiomatic system as a normative one. (This is analogous to the well-known axiomatizations of the metric d_{KSB} by Kemeny and Bogart.) It seems to us that the intuitive appeal of the D^μ functions (or lack thereof) as dissimilarity metrics is plain, and is discussed in Section 1. Instead, our goal is to dissect these metrics here, and uncover some of their structural properties that are unique to this class. As we shall see later, this not only will make performing computations with D^μ maps easier, but also will highlight the geometry of the semimetric space $(\mathbb{A}(X), D^\mu)$.

Let d be a semimetric on $\mathbb{A}(X)$. The first axiom we impose on d says simply that if an acyclic order is in-between two acyclic orders on X, say, $\succsim$ and $\trianglerighteq$, then that order must lie on the metric segment between $\succsim$ and $\trianglerighteq$ relative to d.[16] That is:

---

[15]For instance, let X consist of the 2-vectors $x = (0, 5)$, $y = (5, 0)$ and $z = (6, 1)$, and let $\succsim$ be the coordinatewise ordering on X. Then, ind($\succsim$) contains all elements of $X \times X$ except (y, z) and (z, y), while $\sim$ equals Δ_X.

[16]A *metric segment* between two points x and y in a semimetric space (E, ρ) is defined as $\{z \in E : d(x, y) = d(x, z) + d(z, y)\}$. In the context of any normed linear space, this notion coincides with that of a line segment.

Axiom 1. *For any* $\succsim$, $\succsim_0$ *and* $\trianglerighteq$ *in* $\mathbb{A}(X)$ *such that* $\succsim_0$ *is in-between* $\succsim$ *and* $\trianglerighteq$, *we have*

$$d(\succsim, \trianglerighteq) = d(\succsim, \succsim_0) + d(\succsim_0, \trianglerighteq).$$

We may, of course, equivalently state this axiom in the following way which is easier to check:

Axiom 1'. *For any* $\succsim$, $\succsim_0$, *and* $\trianglerighteq$ *in* $\mathbb{A}(X)$ *such that* $\succsim \to \succsim_0 \twoheadrightarrow \trianglerighteq$, *we have*

$$d(\succsim, \trianglerighteq) = d(\succsim, \succsim_0) + d(\succsim_0, \trianglerighteq).$$

Geometrically speaking, then, what Axiom 1 establishes is that the metric segment between $\succsim$ and $\trianglerighteq$ (relative to the semimetric d) is precisely the set of all preference relations in $\mathbb{A}(X)$ that are in-between $\succsim$ and $\trianglerighteq$. Intutively, when we perturb a given preference $\succsim$, we may be moving in any one direction in the space $\mathbb{A}(X)$. But if we perturb it toward $\trianglerighteq$ – note that, at the level of its definition, there is noting geometric about this operation – we wish the metric d to recognize this as really moving in the direction of $\trianglerighteq$. In the context of metric geometry, the only way we can say this is by locating such a perturbation in the metric segment between $\succsim$ and $\trianglerighteq$. This is the geometric content of Axiom 1', and hence of Axiom 1.

From a more operational standpoint, we may also think of Axiom 1 as an *additivity* property. For instance, when $\succsim \to \succsim_0 \twoheadrightarrow \trianglerighteq$, we know that $\succsim_0$ and $\trianglerighteq$ are "more similar" than $\succsim$ and $\trianglerighteq$ are, so a metric d that captures the dissimilarity of acyclic orders should certainly declare that $d(\succsim, \trianglerighteq) > d(\succsim_0, \trianglerighteq)$. Axiom 1' says further that the "excess dissimilarity" of $\succsim$ and $\trianglerighteq$ additively decomposes into the dissimilarity of $\succsim$ and $\succsim_0$ and that of $\succsim_0$ and $\trianglerighteq$. As such, Axiom 1' (hence Axiom 1) are not only duly compatible with how we view the notion of one-step perturbations (and hence the concept of being in-between), but it also brings a mathematically convenient structure for accounting the effects of such perturbations.[17]

Axiom 1 determines the metric segments between any two preference relations, but it does not say anything about the lengths of these segments (which would then determine d uniquely). Intuitively, to find these lengths all we need to do is to choose the numbers to assign as the distances between any two *adjacent* preferences on a metric segment.

For any distinct $a, b \in X$, we define $\succsim_{ab}$ and $\succsim_{ab}^{+}$ as the partial orders on X whose asymmetric parts are given as

$$\succ_{ab} \; := (X \backslash \{a, b\}) \times \{a, b\}$$

and

$$\succ_{ab}^{+} \; := \; \succ_{ab} \sqcup \{(a, b)\}.$$

In words, $\succsim_{ab}$ ranks every alternative other than a and b strictly above both a and b, making no other pairwise comparisons (including that between a and b). In turn, $\succsim_{ab}^{+}$ is the same relation as $\succsim_{ab}$ except that it ranks a strictly higher than b. (See Figure 6 for the Hasse diagrams of these partial orders in the case where X has six elements.)

[17] There are many papers in the literature on metrics for preference relations in which such additivity axioms are used; see, for instance, [5, 6, 15]. The difference of Axiom 1 from its predecessors lies in the way we defined the notion of one-step perturbations, and hence the concept of being in-between.

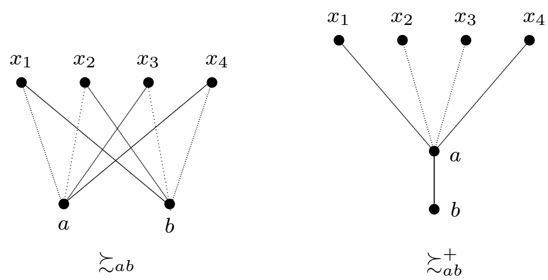

Figure 6

The following axiom is a neutrality property that posits that the distance between $\succsim_{ab}$ and $\succsim_{ab}^+$ is independent of both a and b, and normalizes this distance to 1. Among the D^μ semimetrics, it is satisfied only by D.

Axiom 2. *$d(\succsim_{ab}, \succsim_{ab}^+) = 1$ for every distinct $a, b \in X$.*

We are now ready to take up the problem of assigning distances to adjacent preferences on the line segment between any two given preferences. To this end, let us define

$$N(b, \succsim) := |\{x \in X \backslash \{b\} : \text{ not } x \succ b\}|$$

for any $b \in X$ and $\succsim \in \mathbb{A}(X)$. Thus $N(b, \succsim)$ is the number of elements of $X \backslash \{b\}$ that are not ranked strictly higher than b by $\succsim$.

To understand the significance of this number, take any $\succsim \in \mathbb{A}(X)$ and any $a, b \in X$ with $a \succ b$. Put $\succsim_0 := \succsim \ominus (a, b)$. Then, there are menus S for which b is $\succsim_0$-maximal – that is, b is a "choice" from S for an individual with preferences $\succsim_0$ – but it is not $\succsim$-maximal. This happens precisely for those $S \subseteq X$ such that

$$S = \{a, b\} \sqcup T \quad \text{ for some } T \subseteq N(b, \succsim). \tag{7}$$

Moreover, on each such menu, the set of "choices" on the basis of $\succsim$ and $\succsim_0$ differ from each other by $\{b\}$ just as the set of "choices" on the basis of $\succsim_{ab}$ and $\succsim_{ab}^+$ differ from each other by $\{b\}$. Consequently, per such menu, it makes sense to deem the dissimilarity between $\succsim$ and $\succsim_0$ as the same as that between $\succsim_{ab}$ and $\succsim_{ab}^+$, at least insofar as we wish to capture the dissimilarity of preference relations on the basis of what they declare maximal in various menus. As there are $2^{N(b,\succsim)}$ many menus that satisfy (7) (by definition of $N(b, \succsim)$), therefore, a consistent assignment of a "distance" between $\succsim$ and $\succsim_0$ would be $2^{N(b,\succsim)} d(\succsim_{ab}, \succsim_{ab}^+)$.

We can reason analogously when $(a, b) \in \text{Inc}(\succ)$ and $\succsim_0$ equals, instead, $\succsim \oplus (a, b)$. In this case, a pivotal menu S would be a subset of X such that $a \in S$ and $b \in M(S, \succsim)$. This happens for those $S \subseteq X$ such that

$$S = \{b\} \sqcup T \quad \text{ for some } T \subseteq N(b, \succsim) \text{ with } a \in T.$$

By definition of $N(b, \succsim)$ there are exactly $2^{N(b,\succsim)-1}$ many such menus, so reasoning as in the previous paragraph, we arrive at the conclusion that a consistent assignment of a "distance" between $\succsim$ and $\succsim_0$ is $2^{N(b,\succsim)-1} d(\succsim_{ab}, \succsim_{ab}^+)$.

These considerations prompt:

Axiom 3. *For any $\succsim \in \mathbb{A}(X)$ and $a, b \in X$, if a and b are not $\succ$-comparable,*

$$d(\succsim, \succsim \oplus (a, b)) = 2^{N(b,\succsim)-1} d(\succsim_{ab}, \succsim_{ab}^+),$$

and if $a \succ b$,

$$d(\succsim, \succsim \ominus(a,b)) = 2^{N(b,\succsim)} d(\succsim_{ab}, \succsim_{ab}^{+}).$$

Our final axiom allows us to deal with indifferences, and is very basic. The notion of "dissimilarity" for preferences (acyclic orders) that we focus on in this paper stems from the dissimilarity of the sets of choices that these preferences induce on menus (subsets of X). And, as usual, we model all potential choices of an individual with a given preference relation on a menu S as the set of all maximal elements of S relative to that preference. But maximal elements of a set with respect to a binary relation depends only on the asymmetric part of that relation. That is, the maximal subsets of any $S \subseteq X$ relative to two acyclic orders on X with the same asymmetric part are identical. Thus:

Axiom 4. *For any* $\succsim, \trianglerighteq \in \mathbb{A}(X)$ *with* $\succ = \triangleright$, *we have* $d(\succsim, \trianglerighteq) = 0$.

In the vast majority of the literature on distance functions on preference relations, it is assumed that the preference relations under consideration are partial orders. In that setup, or more generally if we wish to define a metric on the set of all antisymmetric acyclic orders on X, Axiom 4 is vacuously satisfied.

3.3. **Characterization Theorems.** Let $\succsim$ and $\trianglerighteq$ be two acyclic orders on X. By Theorem 2.2, we may determine a chain of one-step perturbations that transform $\succsim$ into $\trianglerighteq$, while Axiom 1 allows us to find the distance between $\succsim$ and $\trianglerighteq$ by summing up the distances between each consecutive perturbations in this chain. In turn, Axiom 3 allows us to compute these distances in terms of rather special partial orders (of the form $\succsim_{ab}$ and $\succsim_{ab}^{+}$). In addition, we can compute these distances exactly by using Axioms 1 and 3 jointly.

While there are some technicalities to sort out, this strategy leads to the following characterization theorem:

Theorem 3.1. *For any nonempty finite set* X, *a semimetric* $d : \mathbb{A}(X) \times \mathbb{A}(X) \to [0, \infty)$ *satisfies Axioms 1, 3 and 4 if, and only if,* d *is the* μ*-top-difference semimetric for some positive measure* μ *on* 2^X.

Adding Axiom 2 to the mix yields:

Theorem 3.2. *For any nonempty finite set* X, *a semimetric* $d : \mathbb{A}(X) \times \mathbb{A}(X) \to [0, \infty)$ *satisfies Axioms 1-4 if, and only if,* d *is the top-difference semimetric.*

The remaining part of this section is devoted to proving Theorem 3.1; the proof of Theorem 3.2 will be contained within that of Theorem 3.1. To prove the "if" part of this theorem, we will use the following fact:

Lemma 3.3. *For any* $\succsim$, $\succsim_0$, $\trianglerighteq \in \mathbb{A}(X)$ *with* $\succsim \to \succsim_0 \twoheadrightarrow \trianglerighteq$, *and* $S \subseteq X$, *the sets* $M(S, \succsim) \triangle M(S, \succsim_0)$ *and* $M(S, \succsim_0) \triangle M(S, \trianglerighteq)$ *are disjoint, and their union equals* $M(S, \succsim) \triangle M(S, \trianglerighteq)$.

Proof. There are two cases to consider. In the first case, there exist $a, b \in X$ such that $\succsim_0 = \succsim \oplus(a, b)$, $(a, b) \in \mathrm{Inc}(\succ)$ and $a \rhd b$. In this case, by definition of $\succsim_0$, we have $a \succ_0 b$. Note that if either $a \notin S$ or $b \notin M(S, \succsim)$, we have $M(S, \succsim) = M(S, \succsim_0)$, so there is nothing to prove. Let us then assume that $a \in S$ and $b \in M(S, \succsim)$. Since $a \succ_0 b$ and $a \rhd b$, we then have $M(S, \succsim) = M(S, \succsim_0) \sqcup \{b\}$ while b belongs to neither $M(S, \succsim_0)$ nor $M(S, \trianglerighteq)$. It follows that $M(S, \succsim) \triangle M(S, \succsim_0) = \{b\}$ while $b \in M(S, \succsim) \triangle M(S, \trianglerighteq)$. But then

$$\begin{aligned} M(S, \succsim_0) \triangle M(S, \trianglerighteq) &= (M(S, \succsim) \backslash \{b\}) \triangle M(S, \trianglerighteq) \\ &= (M(S, \succsim) \triangle M(S, \trianglerighteq) \backslash \{b\}. \end{aligned}$$

The two assertions of the lemma follow from these calculations.

In the second case, there exist $a, b \in X$ such that $\succsim_0 = \succsim \ominus(a, b)$, $a \succ b$, not $a \rhd b$ and (2) holds. If either $a \notin S$ or $b \notin M(S, \succsim_0)$, we have $M(S, \succsim) = M(S, \succsim_0)$, so there is nothing to prove. We thus assume $a \in S$ and $b \in M(S, \succsim_0)$. Then, since $a \succ b$, b does not belong to $M(S, \succsim)$, and it readily follows from the definition of $\succsim_0$ that $M(S, \succsim_0) = M(S, \succsim) \sqcup \{b\}$. On the other hand, we now have $b \in M(S, \trianglerighteq)$. (Otherwise, there exists an $x \in S$ with $x \rhd b$, so (2) implies $x \succ b$. Given that $a \rhd b$ is not true, x must be distinct from a, so we must conclude that b is not $\succsim_0$-maximal in S, a contradiction.) This implies $b \in M(S, \succsim) \triangle M(S, \trianglerighteq)$, and therefore,

$$\begin{aligned} M(S, \succsim_0) \triangle M(S, \trianglerighteq) &= (M(S, \succsim) \sqcup \{b\}) \triangle M(S, \trianglerighteq) \\ &= (M(S, \succsim) \triangle M(S, \trianglerighteq) \backslash \{b\}. \end{aligned}$$

The two assertions of the lemma follow from these calculations. □

Let μ be any positive measure in 2^X. An obvious application of Lemma 3.3 shows that D^μ satisfies Axiom 1', and by induction, Axiom 1. On the other hand, for any distinct $a, b \in X$, we have $M(S, \succsim_{ab}) = M(S, \succsim_{ab}^+)$ for every $S \subseteq X$ distinct from $\{a, b\}$, while $M(\{a, b\}, \succsim_{ab}) = \{a, b\}$ and $M(\{a, b\}, \succsim_{ab}^+) = \{a\}$, so we obviously have

$$D^\mu(\succsim_{ab}, \succsim_{ab}^+) = \mu\{b\}. \tag{8}$$

This shows that D satisfies Axiom 2. In turn, to show that D^μ satisfies Axiom 3, take any $\succsim \in \mathbb{A}(X)$ and $a, b \in X$. Assume first that a and b are not $\succ$-comparable, and put $\succsim_0 = \succsim \oplus(a, b)$. As we have shown in the proof of Lemma 3.3, $M(S, \succsim) \triangle M(S, \succsim_0) = \varnothing$ if either $a \notin S$ or $b \notin M(S, \succsim_0)$, while $M(S, \succsim) \triangle M(S, \succsim_0) = \{b\}$ if $a \in S$ and $b \in M(S, \succsim_0)$. Therefore, where $\mathcal{S} := \{S \in 2^X : a \in S \text{ and } b \in M(S, \succsim_0)\}$, we have

$$D^\mu(\succsim, \succsim_0) = \sum_{S \in \mathcal{S}} \mu(\{b\}) = |\mathcal{S}|\, \mu(\{b\}). \tag{9}$$

But, since $a \succ b$ is false, we have $|\mathcal{S}| = 2^{N(b, \succsim)-1}$, and combining this with (8) and (9), we find $D^\mu(\succsim, \succsim_0) = 2^{N(b, \succsim)-1} \mu\{b\} = 2^{N(b, \succsim)-1} D^\mu(\succsim_{ab}, \succsim_{ab}^+)$, as desired. That $D^\mu(\succsim, \succsim \ominus(a, b)) = 2^{N(b, \succsim)} D^\mu(\succsim_{ab}, \succsim_{ab}^+)$ when $a \succ b$ is analogously proved. Finally, it is plain that D^μ satisfies Axiom 4. We conclude that D^μ satisfies Axioms 1-4.

We now proceed to prove the "only if" part of Theorem 3.1. First, a preliminary observation:

Lemma 3.4. *Let $d : \mathbb{A}(X) \times \mathbb{A}(X) \to [0, \infty)$ be a semimetric that satisfies Axioms 1 and 3. Then,*

$$d(\succsim_{ab}, \succsim_{ab}^{+}) = d(\succsim_{cb}, \succsim_{cb}^{+}) \quad \text{for every distinct } a, b, c \in X.$$

Proof. Take any distinct $a, b, c \in X$, put $Y := X \backslash \{a, b, c\}$, and consider the partial orders $\succsim$ and $\trianglerighteq$ on X whose asymmetric parts are given as

$$Y \succ \{a, b, c\} \quad \text{and} \quad Y \vartriangleright \{a, c\} \vartriangleright b.$$

(In particular, no two distinct element of Y (if any) are comparable by either $\succsim$ or $\trianglerighteq$.) Then, $\succsim \to \succsim \oplus(a, b) \twoheadrightarrow \trianglerighteq$ so that $d(\succsim, \trianglerighteq) = d(\succsim, \succsim \oplus(a, b)) + d(\succsim \oplus(a, b), \trianglerighteq)$ by Axiom 1'. Now by Axiom 3, $d(\succsim, \succsim \oplus(a, b)) = (2^{2-1})d(\succsim_{ab}, \succsim_{ab}^{+})$. On the other hand, we have

$$\trianglerighteq \ = \ (\succsim \oplus(a, b)) \oplus (c, b),$$

so applying Axiom 3 again yields $d(\succsim \oplus(a, b), \trianglerighteq) = (2^{1-1})d(\succsim_{cb}, \succsim_{cb}^{+})$. Conclusion:

$$d(\succsim, \trianglerighteq) = 2d(\succsim_{ab}, \succsim_{ab}^{+}) + d(\succsim_{cb}, \succsim_{cb}^{+}).$$

But we also have $\succsim \to \succsim \oplus(c, b) \twoheadrightarrow \trianglerighteq$ and $\trianglerighteq = (\succsim \oplus(c, b)) \oplus (a, b)$, so repeating this reasoning yields

$$d(\succsim, \trianglerighteq) = d(\succsim_{ab}, \succsim_{ab}^{+}) + 2d(\succsim_{cb}, \succsim_{cb}^{+}).$$

Combining these two equations gives $d(\succsim_{ab}, \succsim_{ab}^{+}) = d(\succsim_{cb}, \succsim_{cb}^{+})$. □

Now let d be a semimetric on $\mathbb{A}(X)$ that satisfies Axioms 1, 3 and 4. For any $b \in X$, we define $m_b := d(\succsim_{ab}, \succsim_{ab}^{+})$ where $a \in X \backslash \{b\}$. By Lemma 3.4, m_b is well-defined nonnegative real number for each $b \in X$. We define $\mu : 2^X \to [0, \infty)$ by $\mu(\varnothing) := 0$ and $\mu(S) := \sum_{b \in S} m_b$ for every nonempty $S \subseteq X$. Obviously, μ is a positive measure on 2^X (and it is the counting measure if d satisfies Axiom 2.) We will complete our proof by showing that $d = D^\mu$.

Take any $\succsim \in \mathbb{A}(X)$. Then, for any $(a, b) \in \text{Inc}(\succ)$,

$$\begin{aligned} d(\succsim, \succsim \oplus(a, b)) &= 2^{N(b, \succsim)-1} d(\succsim_{ab}, \succsim_{ab}^{+}) \\ &= 2^{N(b, \succsim)-1} \mu(\{b\}) \\ &= D^\mu(\succsim, \succsim \oplus(a, b)), \end{aligned}$$

where the first equality follows from Axiom 3, the second follows from the fact that $\mu(\{b\}) = m_b = d(\succsim_{ab}, \succsim_{ab}^{+})$ for any $a \in X \backslash \{b\}$, and the third was established above at the end of the proof of the "if" part of the theorem. If $a \succ b$, the analogous reasoning would show instead that $d(\succsim, \succsim \ominus(a, b)) = D^\mu(\succsim, \succsim \ominus(a, b))$. Conclusion: d and D^μ have the same value at $(\succsim, \trianglerighteq)$ for every $\succsim, \trianglerighteq \in \mathbb{A}(X)$ where $\trianglerighteq$ is a one-step perturbation of $\succsim$.

Now take any $\succsim$, $\trianglerighteq \in \mathbb{A}(X)$ and assume that the symmetric parts of these relations are the same. If $\trianglerighteq$ is a one-step perturbation of $\succsim$, we know that $d(\succsim, \trianglerighteq) = D^\mu(\succsim, \trianglerighteq)$. Otherwise, we apply Theorem 2.2 to find an integer $n \geq 2$ and $\succsim_0, ..., \succsim_{n-2} \in \mathbb{A}(X)$ such that $\succsim \to \succsim_0 \twoheadrightarrow \trianglerighteq$ and $\succsim_{k-1} \to \succsim_k \twoheadrightarrow \trianglerighteq$ for each $k = 1, ..., n-1$, and $\succsim_{n-1} = \trianglerighteq$. Consequently, applying Axiom 1' inductively,

$$\begin{aligned} d(\succsim, \trianglerighteq) &= d(\succsim, \succsim_0) + \cdots + d(\succsim_{n-2}, \succsim_{n-1}) \\ &= D^\mu(\succsim, \succsim_0) + \cdots + D^\mu(\succsim_{n-2}, \trianglerighteq) \\ &= D^\mu(\succsim, \trianglerighteq) \end{aligned}$$

where the third equality follows from the fact that D^μ satisfies Axiom 1'.

Finally, take any $\succsim$, $\trianglerighteq \in \mathbb{A}(X)$, and define $\succsim^* := \succ \sqcup \triangle_X$ and $\trianglerighteq^* := \triangleright \sqcup \triangle_X$. Then, $\succsim^*, \trianglerighteq^* \in \mathbb{A}(X)$ and $d(\succsim^*, \trianglerighteq^*) = D^\mu(\succsim^*, \trianglerighteq^*)$ by what we have found in the previous paragraph. But, by Axiom 4, $d(\succsim, \succsim^*) = 0 = d(\trianglerighteq, \trianglerighteq^*)$. Since d is a semimetric, therefore,

$$d(\succsim, \trianglerighteq) = d(\succsim, \succsim^*) + d(\succsim^*, \trianglerighteq^*) + d(\trianglerighteq^*, \trianglerighteq) = d(\succsim^*, \trianglerighteq^*) = D^\mu(\succsim^*, \trianglerighteq^*).$$

Since $M(S, \succsim^*) = M(S, \succsim)$ and $M(S, \trianglerighteq^*) = M(S, \trianglerighteq)$ for every $S \subseteq X$, we have $D^\mu(\succsim^*, \trianglerighteq^*) = D^\mu(\succsim, \trianglerighteq)$, and hence obtains $d(\succsim, \trianglerighteq) = D^\mu(\succsim, \trianglerighteq)$. The proof of Theorem 3.1 is now complete.

3.4. **Top-Difference Metrics vs. Weighted KSB Metrics.** As we have noted in Section 1, Can [9] and Hassanzadeh and Milenkovic [18] were motivated by observations such as the one we presented in Example 1.1, and have consequently proposed a class of metrics that consist of weighted forms of the classical Kemeny-Snell metric. It should be noted that these metrics are defined only on $\mathbb{L}(X)$, the set of all *linear* orders on X. Moreover, it is not at all clear how to extend these metrics (axiomatically or even simply by definition) to the domains like $\mathbb{P}(X)$ or $\mathbb{P}^*(X)$. As such, we can make a comparison with these metrics and the μ-top-difference semimetrics only by restricting the domain of the latter to $\mathbb{L}(X)$. (As noted earlier, on this domain, any D^μ acts as a metric.)

Let $n := |X|$, and let Σ denote the set of all permutations σ on $\{1, ..., n\}$ for which there is a $k \in \{1, ..., n-1\}$ with $\sigma(k) = k+1$, $\sigma(k+1) = \sigma(k)$, and $\sigma(i) = i$ for all $i \neq k, k+1$. ([18] refer to such permutations as *adjacent transpositions.*) In what follows, we abuse notation and write $\sigma(x_1, ..., x_n)$ for the n-vector $(x_{\sigma(1)}, ..., x_{\sigma(n)})$ for any $x_1, ..., x_n \in X$ and $\sigma \in \Sigma$.

Next, for any $\succsim \in \mathbb{L}(X)$, let us agree to write $v(\succsim)$ for the n-vector $(x_1, ..., x_n)$ where $x_1 \succ \cdots \succ x_n$. Finally, for any (weight function) $\omega : \Sigma \to [0, \infty)$, we define the real map d_ω on $\mathbb{L}(X) \times \mathbb{L}(X)$ by

$$d_\omega(\succsim, \trianglerighteq) := \min \sum_{i=1}^{k} \omega(\sigma_i)$$

where the minimum is taken over all $k \in \mathbb{N}$ and $\sigma_1, ..., \sigma_k \in \Sigma$ such that $v(\trianglerighteq) = (\sigma_1 \circ \cdots \circ \sigma_k) v(\succsim)$. It is easy to check that this is indeed a metric on $\mathbb{L}(X)$; it is referred to as a *weighted Kendall metric* by [18]. Of course, for the weight function ω that equals 2 everywhere, d_ω becomes precisely the Kemeny-Snell-Bogart metric on $\mathbb{L}(X)$.

The first observation we would like to make here is that if we choose the weight function ω such that $\omega(\sigma) = 2^{n-k}$ for the adjacent transposition σ with $\sigma(k) = k+1$ and $\sigma(k+1) = \sigma(k)$, then d_ω becomes identical to the top-difference metric on $\mathbb{L}(X)$, that is, $d_\omega = D|_{\mathbb{L}(X) \times \mathbb{L}(X)}$ for this special weight function ω. (We omit the straightforward proof.) Second, when $n \geq 3$, there is no weight function ω such that $d_\omega = D^\mu|_{\mathbb{L}(X) \times \mathbb{L}(X)}$ unless μ is indeed the counting measure. To see this, pick any measure μ on 2^X such that $\mu(\{x\}) \neq \mu(\{y\})$ for some $x, y \in X$. Now take any $a \in X \backslash \{x, y\}$, and consider the linear orders $\succsim_1, ..., \succsim_4$ on X such that

$$x \succ_1 \cdots \succ_1 a \succ_1 y \quad \text{and} \quad x \succ_2 \cdots \succ_2 y \succ_2 a,$$

and

$$y \succ_3 \cdots \succ_3 x \succ_3 a, \quad \text{and } y \succ_4 \cdots \succ_4 a \succ_4 x,$$

with the understanding that the unspecified parts of all of these linear orders agree. Then, it is plain that $d_\omega(\succsim_1, \succsim_2) = d_\omega(\succsim_3, \succsim_4)$ for any $\omega : \Sigma \to [0, \infty)$. By contrast, $D^\mu(\succsim_1, \succsim_2) = \mu(\{a, y\}) \neq \mu(\{a, x\}) = D^\mu(\succsim_3, \succsim_4)$, which shows that D^μ is distinct from d_ω no matter how we may choose the weight function ω. We proved:

Proposition 3.5. *For any finite set X with $|X| \geq 3$, the only μ-top-difference metric which is also a weighted Kendall metric is the top-difference metric on $\mathbb{L}(X)$.*

Thus, not only does the metrization approach we develop here applies well beyond $\mathbb{L}(X)$, even on this space it is quite distinct from those of [9] and [18]. There is only one exception to this, namely, the top-difference metric D on $\mathbb{L}(X)$. This metric is the only one that lies in the intersection of the D^μ class and the class of metrics introduced by [9] and [18].[18] This observation further singles out this semimetric as the most important member of the class of D^μ metrics, and provides motivation for its further investigation.[19]

3.5. **On the Computational Complexity for D^μ.** While the intuition behind the D^μ metrics appears convincing, and it is reinforced by Theorems 3.1 and 3.2, the computation of distances between two acyclic preferences $\succsim$ and $\trianglerighteq$ on X according to any one of these metrics require one compute the symmetric difference between $M(S, \succsim)$ and $M(S, \trianglerighteq)$ for all subsets S of X. As this set is empty whenever $|S| \leq 1$, this means we have to compute $M(S, \succsim) \triangle M(S, \trianglerighteq)$ for $2^{|X|} - |X| - 1$ many subsets of X. As $|X|$ gets larger, this becomes a computationally daunting task. This is in stark contrast with the computation of distances relative to the Kemeny-Snell-Bogart metric which requires at most polynomial time with respect to the size of X.

Fortunately, there is a more efficient way of computing $D^\mu(\succsim, \trianglerighteq)$ for any given $\succsim, \trianglerighteq \in \mathbb{A}(X)$ which we now explore. For any $S \subseteq X$, let us first write

$$\triangle_S(\succsim, \trianglerighteq) := M(S, \succsim) \triangle M(S, \trianglerighteq)$$

to simplify our notation. Then, for any fixed positive measure μ on 2^X, we have

$$\begin{aligned} D^\mu(\succsim, \trianglerighteq) &= \sum_{S \subseteq X} \mu(\triangle_S(\succsim, \trianglerighteq)) \\ &= \sum_{S \subseteq X} \sum_{x \in S} \mu(\{x\}) \mathbf{1}_{\triangle_S(\succsim, \trianglerighteq)}(x) \\ &= \sum_{x \in X} \left(\sum_{\substack{S \subseteq X \\ S \ni x}} \mathbf{1}_{\triangle_S(\succsim, \trianglerighteq)}(x) \right) \mu(\{x\}) \end{aligned}$$

In other words,

$$D^\mu(\succsim, \trianglerighteq) = \sum_{x \in X} \theta_x(\succsim, \trianglerighteq) \mu(\{x\})$$

where $\theta_x(\succsim, \trianglerighteq)$ is the number of all subsets S of X such that $x \in \triangle_S(\succsim, \trianglerighteq)$.

[18] We should note that the special weight function that yields D within the class of all weighted Kendall metrics is not mentioned in either [9] or [18]. These papers do not provide an axiomatization for this semimetric even on $\mathbb{L}(X)$.

[19] Precisely the same observation applies to the class of metrics proposed by Baldiga and Green [2]. This class is also defined only on $\mathbb{L}(X)$, and it contains $D^\mu|_{\mathbb{L}(X) \times \mathbb{L}(X)}$ iff μ is the counting measure.

Let us now fix any $x \in X$, and calculate $\theta_x(\succsim, \trianglerighteq)$. To this end, let us define the following three sets:

$$A_x(\succsim, \trianglerighteq) := \{a \in X\backslash\{x\} : \text{ not } a \succ x \text{ and not } a \vartriangleright x\},$$

and

$$B_x(\succsim, \trianglerighteq) := \{a \in X\backslash\{x\} : a \succ x \text{ but not } a \vartriangleright x\},$$

and

$$C_x(\succsim, \trianglerighteq) := \{a \in X\backslash\{x\} : a \vartriangleright x \text{ but not } a \succ x\}.$$

We denote the cardinality of the first of these sets by $\alpha_x(\succsim, \trianglerighteq)$. Notice first that $x \in M(S, \succsim)\backslash M(S, \trianglerighteq)$ iff $S = \{x\} \sqcup K \sqcup L$ for some $K \subseteq A_x(\succsim, \trianglerighteq)$ and some *nonempty* $L \subseteq C_x(\succsim, \trianglerighteq)$. There are exactly $2^{\alpha_x(\succsim,\trianglerighteq)}(2^{|C_x(\succsim,\trianglerighteq)|} - 1)$ many such sets. On the other hand, by the same logic, there are $2^{\alpha_x(\succsim,\trianglerighteq)}(2^{|B_x(\succsim,\trianglerighteq)|} - 1)$ many subsets S of X such that $x \in M(S, \trianglerighteq)\backslash M(S, \succsim)$. It follows that

$$\theta_x(\succsim, \trianglerighteq) = 2^{\alpha_x(\succsim,\trianglerighteq)}(2^{|B_x(\succsim,\trianglerighteq)|} + 2^{|C_x(\succsim,\trianglerighteq)|} - 2).$$

Next, notice that $A_x(\succsim, \trianglerighteq) \sqcup B_x(\succsim, \trianglerighteq) = \{a \in X\backslash\{x\} : \text{not } a \vartriangleright x\}$, whence

$$\alpha_x(\succsim, \trianglerighteq) + |B_x(\succsim, \trianglerighteq)| = n - \left|x^{\uparrow,\vartriangleright}\right| - 1$$

where $n := |X|$, and as we defined in Section 2.1, $x^{\uparrow,\vartriangleright}$ is the principal ideal of x with respect to $\vartriangleright$. Of course, the analogous reasoning shows that $\alpha_x(\succsim, \trianglerighteq) + |C_x(\succsim, \trianglerighteq)| = n - \left|x^{\uparrow,\succ}\right| - 1$ as well. Consequently,

$$\theta_x(\succsim, \trianglerighteq) = 2^{n-\left|x^{\uparrow,\vartriangleright}\right|-1} + 2^{n-\left|x^{\uparrow,\succ}\right|-1} - 2^{\alpha_x(\succsim,\trianglerighteq)+1}.$$

Combining the computations of the previous two paragraphs yields an alternative method of calculating the distance between $\succsim$ and $\trianglerighteq$ with respect to D^μ:

$$D^\mu(\succsim, \trianglerighteq) = \sum_{x \in X} \left[2^{n-\left|x^{\uparrow,\vartriangleright}\right|-1} + 2^{n-\left|x^{\uparrow,\succ}\right|-1} - 2^{\alpha_x(\succsim,\trianglerighteq)+1}\right] \mu(\{x\}). \tag{10}$$

This formula does not look particularly appealing at first glance. It is not even clear that it defines a semimetric on $\mathbb{A}(X)$, and it is certainly not intuitive. However, it has a significant computational advantage over the formula we defined D^μ with. Indeed, this formula uses only "local" knowledge about the involved acyclic orders. As a consequence, the computation of the numbers $\left|x^{\uparrow,\vartriangleright}\right|$, $\left|x^{\uparrow,\succ}\right|$ and $\alpha_x(\succsim, \trianglerighteq)$ for each $x \in X$, and hence the above formula, take at most polynomial time with respect to the size of X, which parallels the computational efficiency of the Kemeny-Snell-Bogart metric. Any sort of a program that is primed to compute the values of D^μ should thus utilize (10) instead of (6). The computational superiority of (10) over (6) will be further witnessed in the next subsection.

3.6. **The Distance Between Linear Orders.** The family $\mathbb{L}(X)$ of linear orders on X arises in numerous applications, ranging from voting theory to stable matching, random utility theory, etc.. Indeed, the Kemeny-Snell-Bogart metric is primarily applied on $\mathbb{L}(X)$ (and as such, it is often simply referred to as the *Kemeny-Snell metric*). It is thus natural to ask if there is an easy way of computing the top-difference metric D on $\mathbb{L}(X) \times \mathbb{L}(X)$. (We recall that D is a metric on $\mathbb{L}(X)$, not only a semimetric.) We next provide such a formula by using (10).

Take any $\succsim, \trianglerighteq \in \mathbb{L}(X)$, and put $n := |X|$. Given that $\succsim$ is a linear order, for every $i \in \{0, ..., n-1\}$, there is a unique $x \in X$ such that $\left|x^{\downarrow,\succ}\right| = i$. Moreover, again by linearity, $\left|x^{\downarrow,\succ}\right| = n - \left|x^{\uparrow,\succ}\right| - 1$ for each $x \in X$. It follows that

$$\sum_{x \in X} 2^{n - \left|x^{\uparrow,\succ}\right| - 1} = \sum_{i=0}^{n-1} 2^i = 2^n - 1.$$

Since, analogously, we also have $\sum_{x \in X} 2^{n - \left|x^{\uparrow,\rhd}\right| - 1} = 2^n - 1$, the formula (10) yields

$$D(\succsim, \trianglerighteq) = 2(2^n - 1) - \sum_{x \in X} 2^{\alpha_x(\succsim, \trianglerighteq) + 1}.$$

Next, notice that $\alpha_x(\succsim, \trianglerighteq)$ is none other than the number of all elements of X that are strictly below x with respect to both $\succsim$ and $\trianglerighteq$ (again because $\succsim$ and $\trianglerighteq$ are linear orders on X). Consequently, we arrive at

$$D(\succsim, \trianglerighteq) = 2(2^n - 1) - \sum_{x \in X} 2^{\left|x^{\downarrow,\succ} \cap x^{\downarrow,\rhd}\right| + 1}.$$

This shows that to find the distance between two linear orders on X, all one has to do is to count the elements in the intersection of the principal filters of each $x \in X$ with respect to $\succsim$ and $\trianglerighteq$. This is very efficient, as it allows us to work with the orders $\succsim$ and $\trianglerighteq$ separately.

4. Diameter of the Preference Space $(\mathbb{A}(X), D)$

To get a better sense of the "distance" between two preference relations in practice, one should really have a basic benchmark. In particular, it may be useful to know the *diameter* of the space of preferences one is interested in with respect to the semimetric at hand. In this section we thus attempt to get some simple lower estimates for the diameter of $\mathbb{A}(X)$ and $\mathbb{P}(X)$ with respect to the top-difference semimetric D. (We denote the diameter operator relative to D by $\mathrm{diam}_D(\cdot)$.)

Let us denote the cardinality of X by n; recall that $n \geq 2$. The diameter problem is easily treated in the case of linear orders. Indeed, for any $\succsim$, $\succsim' \in \mathbb{L}(X)$, the cardinality of $M(S, \succsim) \triangle M(S, \succsim')$ is at most 2 for any $S \subseteq X$ with at least two elements. Therefore, the largest possible value for $D(\succsim, \succsim')$ is 2 times the number of all $S \subseteq X$ with $|S| \geq 2$, namely, $2(2^n - n - 1)$. But if we enumerate X as $\{x_1, ..., x_n\}$, and choose $\succsim$ and $\succsim'$ orthogonally to each other as $x_1 \succ \cdots \succ x_n$ and $x_n \succ' \cdots \succ' x_1$, then $|M(S, \succsim) \triangle M(S, \succsim')| = 2$ for all $S \subseteq X$ with $|S| \geq 2$. Thus:

$$\mathrm{diam}_D(\mathbb{L}(X)) = 2(2^n - n - 1). \tag{11}$$

To put this number in some perspective, we report its value in the table below for the first nine values of n, next to the cardinality $n!$ of $\mathbb{L}(X)$.

The situation is more complicated for total preorders. To examine this case, we fix any $m \in \{1, ..., n-1\}$, and consider the total preorders $\succsim$ and $\succsim'$ on X such that

$$x_1 \sim \cdots \sim x_m \succ x_{m+1} \succ \cdots \succ x_n$$

and

$$x_{m+1} \sim' \cdots \sim' x_n \succ' x_1 \succ' \cdots \succ' x_m.$$

Now let $A := \{x_1, ..., x_m\}$ and $B := \{x_{m+1}, ..., x_n\}$, and note that

$$|M(S, \succsim) \triangle M(S, \succsim')| = \begin{cases} |S| - 1, & \text{if } S \subseteq A \text{ or } S \subseteq B, \\ |S|, & \text{otherwise} \end{cases}$$

for any $S \subseteq X$. Where $\mathcal{S} := \{S \in 2^X \backslash \{\varnothing\} : S \cap A \neq \varnothing \neq S \cap B\}$, we thus have

$$\begin{aligned} D(\succsim, \succsim') &= \sum_{S \in \mathcal{S}} |S| + \sum_{\varnothing \neq S \subseteq A} (|S| - 1) + \sum_{\varnothing \neq S \subseteq B} (|S| - 1) \\ &= \sum_{\varnothing \neq S \subseteq X} |S| - \sum_{\varnothing \neq S \subseteq A} 1 - \sum_{\varnothing \neq S \subseteq A} 1 \\ &= \sum_{k=1}^{n} k \binom{n}{k} - \left|2^A \backslash \{\varnothing\}\right| - \left|2^B \backslash \{\varnothing\}\right| \\ &= \sum_{k=1}^{n} k \binom{n}{k} + 2 - 2^m - 2^{n-m}. \end{aligned}$$

It is readily checked that $t \mapsto 2^t + 2^{n-t}$ is a symmetric and strictly convex function on $[0, n]$; this function attains its unique global minimum at $\frac{n}{2}$. It follows that the map $m \mapsto 2^m + 2^{m-t}$ achieves its minimum on $\{0, ..., m\}$ at $\lfloor \frac{n}{2} \rfloor$. Combining this fact with the calculation above, and recalling that $\sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}$ (which is easily verified by induction on n) and $\lceil \frac{n}{2} \rceil = n - \lfloor \frac{n}{2} \rfloor$, we find that $n2^{n-1} + 2 - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lceil \frac{n}{2} \rceil}$ is a lower bound for $\text{diam}_D(\mathbb{P}_{\text{total}}(X))$. In our next result we prove that this lower bound is actually attained.

Theorem 4.1. *Let X be a finite set with $n := |X| \geq 2$. Then,*

$$\text{diam}_D(\mathbb{P}_{\text{total}}(X)) = n2^{n-1} + 2 - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lceil \frac{n}{2} \rceil}. \tag{12}$$

Proof. Let us begin by noting that for $n = 2$ and $n = 3$, it is readily checked that $\text{diam}_D(\mathbb{L}(X)) = \text{diam}_D(\mathbb{P}_{\text{total}}(X))$ and that the right-hand sides of (11) and (12) are the same. As this observation readily yields the present theorem for $n \in \{2, 3\}$, we assume $n \geq 4$ in the rest of the proof.

Now define $\eta : \{1, ..., n-1\} \to (-\infty, 0)$ by $\eta(m) := 2 - 2^m - 2^{n-m}$. We have seen above that $\eta(\lfloor \frac{n}{2} \rfloor) \geq \eta(m)$ for each $m = 1, ..., n-1$, and that

$$\text{diam}_D(\mathbb{P}_{\text{total}}(X)) \geq n2^{n-1} + \eta(\lfloor \tfrac{n}{2} \rfloor).$$

To prove the converse inequality, we take any total preorders $\succsim$ and $\succsim'$ on X. We must show that $D(\succsim, \succsim') \leq n2^{n-1} + \eta(\lfloor \frac{n}{2} \rfloor)$.

Let us first assume that there is at least one element that is maximal in X with respect to both $\succsim$ and $\succsim'$. Let $\mathcal{A}$ stand for the set of all subsets of X that contain this element, and note that $|\mathcal{A}| = 2^{n-1} = -\eta(1)$. Then, $|M(S, \succsim) \triangle M(S, \succsim')|$ is at most $|S| - 1$ for every $S \in \mathcal{A}$ while it is trivially less than $|S|$ for any $S \subseteq X$. Consequently,

$$\begin{aligned} D(\succsim, \succsim') &\leq \sum_{S \in \mathcal{A}} (|S| - 1) + \sum_{S \in 2^X \backslash \mathcal{A}} |S| \\ &= \sum_{S \subseteq X} |S| - |\mathcal{A}| \\ &= n2^{n-1} + \eta(1) \\ &\leq n2^{n-1} + \eta(\lfloor \tfrac{n}{2} \rfloor), \end{aligned}$$

as desired.[20]

[20] The third equality here holds because $\sum_{S \subseteq X} |S| = \sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}$.

It remains to consider the case $M(X,\succsim)\cap M(X,\succsim')=\varnothing$. There are two possibilities to consider in this case. First, assume that $M(X,\succsim)\sqcup M(X,\succsim')=X$. In this case, we put $m:=|M(X,\succsim)|$, and note that $|M(X,\succsim')|=n-m$. Let $\mathcal{A}$ stand for the set of all nonempty subsets S of X such that either $S\subseteq M(X,\succsim)$ or $S\subseteq M(X,\succsim')$. Since $M(X,\succsim)$ and $M(X,\succsim')$ are disjoint, we have $|\mathcal{A}|=(2^m-1)+(2^{n-m}-1)=-\eta(m)$. On the other hand, again, $|M(S,\succsim)\triangle M(S,\succsim')|\leq|S|-1$ for every $S\in\mathcal{A}$. Therefore, carrying out the same calculation we have done in the previous paragraph yields $D(\succsim,\succsim')\leq n2^{n-1}+\eta(m)\leq n2^{n-1}+\eta(\lfloor\frac{n}{2}\rfloor)$, as desired.

The only remaining case is where $M(X,\succsim)\cap M(X,\succsim')=\varnothing$ and $M(X,\succsim)\sqcup M(X,\succsim')\neq X$. In this case, to simplify our notation, we put $A:=M(X,\succsim)$, $B:=M(X,\succsim')$ and $C=X\backslash(A\sqcup B)$. Let $m_1:=|A|$, $m_2:=|B|$, and note that $|C|=n-m_1-m_2>0$. Next, we define $\mathcal{A}$ exactly as in the previous paragraph, and note that $|\mathcal{A}|=(2^{m_1}-1)+(2^{m_2}-1)$ and $|M(S,\succsim)\triangle M(S,\succsim')|\leq|S|-1$ for every $S\in\mathcal{A}$. Finally, we define

$$\mathcal{B}:=\{S\in 2^X : S\cap A\neq\varnothing,\ S\cap B\neq\varnothing \text{ and } S\cap C\neq\varnothing\}.$$

Then,

$$\begin{aligned}D(\succsim,\succsim') &\leq \sum_{S\in\mathcal{A}}(|S|-1)+\sum_{S\in\mathcal{B}}(|S|-|S\cap C|)+\sum_{S\in 2^X\backslash(\mathcal{A}\sqcup\mathcal{B})}|S| \\ &= \sum_{S\subseteq X}|S|-|\mathcal{A}|-\sum_{S\in\mathcal{B}}|S\cap C| \\ &= n2^{n-1}-(2^{m_1}-1)-(2^{m_2}-1)-\sum_{S\in\mathcal{B}}|S\cap C|.\end{aligned}$$

On the other hand, by definition of $\mathcal{B}$,

$$\begin{aligned}\sum_{S\in\mathcal{B}}|S\cap C| &= (2^{m_1}-1)(2^{m_2}-1)\sum_{k=1}^{n-m_1-m_2}k\binom{n-m_1-m_2}{k} \\ &= (n-m_1-m_2)(2^{m_1}-1)(2^{m_2}-1)2^{n-m_1-m_2-1}.\end{aligned}$$

If $n-m_1-m_2=1$, therefore,

$$\begin{aligned}(2^{m_1}-1)+(2^{m_2}-1)+\sum_{S\in\mathcal{B}}|S\cap C| &= (2^{m_1}-1)+(2^{m_2}-1)+(2^{m_1}-1)(2^{m_2}-1) \\ &= 2^{n-1}-1 \\ &\geq 2^{n-2}+2 \\ &= -\eta(2).\end{aligned}$$

(The inequality here follows because $n\geq 4$ and $2^{t-1}-2^{t-2}-3\geq 0$ for every $t\geq 4$.[21]) If, on the other hand, $n-m_1-m_2\geq 2$, we have

$$\begin{aligned}\sum_{S\in\mathcal{B}}|S\cap C| &\geq 2(2^{m_1}-1)(2^{m_2}-1)2^{n-m_1-m_2-1} \\ &\geq 2^{m_1-1}2^{m_2-1}2^{n-m_1-m_2} \\ &= 2^{n-2}.\end{aligned}$$

[21]This follows from the fact that the map $t\mapsto 2^{t-1}-2^{t-2}-3$ is (strictly) increasing on $[4,\infty)$ and its value at 4 is positive.

(Here we use the fact that $2^t - 2^{t-1} - 1 \geq 0$ for every $t \geq 1$.) Thus, again, we find

$$\begin{aligned}(2^{m_1}-1)+(2^{m_2}-1)+\sum_{S\in\mathcal{B}}|S\cap C| &\geq 2^{m_1}+2^{m_2}-2+2^{n-2}\\ &\geq 4-2+2^{n-2}\\ &\geq 2+2^{n-2}\\ &= -\eta(2).\end{aligned}$$

Returning to the computation of $D(\succsim,\succsim')$, we then get

$$\begin{aligned}D(\succsim,\succsim') &\leq n2^{n-1}-(2^{m_1}-1)-(2^{m_2}-1)-\sum_{S\in\mathcal{B}}|S\cap C|\\ &\leq n2^{n-1}+\eta(2)\\ &\leq n2^{n-1}+\eta(\lfloor \tfrac{n}{2}\rfloor).\end{aligned}$$

The proof of Theorem 4.1 is now complete. □

For any integer $n \geq 2$, let us denote the number of total preorders on the n-element set X by $p(n)$. It is a well known combinatorial fact that this number can be computed as

$$p(n)=\sum_{k=0}^{n} k!S(n,k)$$

where $S(n,k)$ is the number of ways an n-element set can be partitioned into k many nonempty sets; these numbers are known as the *Stirling numbers of the second kind*. Table 1 provides a comparison between $p(n)$ and the D-diameter of $\mathbb{P}_{\text{total}}(X)$ up to $n = 10$.

	$\operatorname{diam}_D(\mathbb{P}_{\text{total}}(X))$	$p(n)$	$\operatorname{diam}_D(\mathbb{L}(X))$	$n!$
$n=2$	2	3	2	2
$n=3$	8	13	8	6
$n=4$	26	75	22	24
$n=5$	70	541	52	120
$n=6$	178	4,683	114	720
$n=7$	426	47,293	240	5,040
$n=8$	994	545,835	494	40,320
$n=9$	2,258	7,087,261	1,004	362,880
$n=10$	5,058	102,247,563	2,026	3,628,800

Table 1

This table suggests that, relative to the size of $\mathbb{P}_{\text{total}}(X)$, the D-diameter of $\mathbb{P}_{\text{total}}(X)$ remains fairly modest, just as in the case of $\mathbb{L}(X)$.

In passing, we note that as an immediate consequence of Theorem 4.1, we have

$$\operatorname{diam}_D(\mathbb{A}(X)) \geq \operatorname{diam}_D(\mathbb{P}(X)) \geq n2^{n-1}+2-2^{\lfloor \frac{n}{2}\rfloor}-2^{\lceil \frac{n}{2}\rceil}.$$

We do not presently know whether or not either of these inequalities hold as equalities.

5. On Best Transitive Approximations

As an acyclic order $\succsim$ on X need not be transitive, a natural problem is to identify the set of all preorders on X that best approximates $\succsim$ in the sense of distance minimizing where we measure distance by D (or by D^μ for some suitable μ). Put differently, the problem is to compute the metric projection of $\succsim$ in $\mathbb{P}(X)$ relative to D (or D^μ). This seems like an interesting problem, and it should eventually be studied from an algorithmic perspective. Here we offer a partial solution to it.

First, we simplify the problem by assuming $\succsim$ is antisymmetric. Second, we concentrate on finding the best approximation to $\succsim$ among all preorders that extend $\succsim$. Recall that a binary relation R on X *extends* $\succsim$ if it is reflexive and satisfies $\succ\ \subseteq R^>$. (That is, an extension R of $\succsim$ is particularly faithful to $\succsim$ in that its ranking of any two $\succsim$-comparable alternatives is identical to the ranking of those alternatives by $\succsim$.) We denote the set of all transitive extensions of $\succsim$ by $\text{Ext}(\succsim)$. For any given positive measure μ on 2^X, a *best transitive extension of* $\succsim$ *relative to* D^μ is any preorder $\succsim^* \in \text{Ext}(\succsim)$ such that

$$D^\mu(\succsim, \succsim^*) = \min\{D^\mu(\succsim, \trianglerighteq) : \ \trianglerighteq\ \in \text{Ext}(\succsim)\}.$$

Fortunately, such extensions have a nice characterization.

Theorem 5.1. *Let μ be a positive measure μ on X. Then, the unique best transitive extension of any antisymmetric $\succsim\ \in \mathbb{A}(X)$ with respect to D^μ is the transitive closure of $\succsim$.*

Before we prove this theorem, we present a simple example that shows that the transitive closure of an antisymmetric acyclic order on X need not be a best approximation among all preorders on X. This witnesses the nontriviality of the general approximation we outlined above.

Example 5.1. Let $X := \{x_1, x_2, x_3, x_4\}$, and let $\succsim$ be the antisymmetric acyclic order on X whose asymmetric part is given as $x_i \succ x_{i+1}$ for $i = 1, 2, 3$. (The transitive closure $\succsim$ is the linear order on X that ranks x_1 the highest, x_2 the second highest, so on.) Consider the reflexive binary relation $\trianglerighteq$ on X whose asymmetric part is given as $x_1 \triangleright x_2$ and $x_3 \triangleright x_4$. Clearly, $\trianglerighteq$ is a partial order on X, although it is not an extension of $\succsim$. Moreover,

$$D(\succsim, \trianglerighteq) = 4 < 5 = D(\succsim, \text{tran}(\succsim)),$$

so $\text{tran}(\succsim)$ is not a best approximation to $\succsim$ *in* $\mathbb{P}(X)$. $\|$

We now turn to the proof of Theorem 5.1. Let us first observe that for any antisymmetric $\succsim\ \in \mathbb{A}(X)$, $\text{tran}(\succsim)$ is a partial order on X that extends $\succsim$.[22]

[22]We use the antisymmetry postulate in Theorem 5.1 only to ensure that $\text{tran}(\succsim)$ is an antisymmetric extension of $\succsim$. As such, Theorem 5.1 applies to all non-antisymmetric $\succsim\ \in \mathbb{A}(X)$ such that $\text{tran}(\succsim) \in \text{Ext}(\succsim)$.

Incidentally, note that $\text{tran}(\succsim)$ need not be an extension of a reflexive relation $\succsim$ on X that is either cyclic or not antisymmetric. To illustrate, let $X := \{a, b, c\}$. If $\succsim$ equals $\Delta_X \sqcup \{(a, b), (b, c), (c, a)\}$, then $\succsim$ is a reflexive and antisymmetric, but not acyclic, binary relation on X, and $\text{tran}(\succsim) = X \times X$ which is not an extension of $\succsim$. On the other hand, if $\succsim$ equals $(X \times X)\backslash\{(c, b)\}$, then $\succsim\ \in \mathbb{A}(X)$ (but $\succsim$ is not antisymmetric) and again $\text{tran}(\succsim) = X \times X$ which is not an extension of $\succsim$.

Lemma 5.2. *For any antisymmetric* $\succsim \in \mathbb{A}(X)$*,* $\text{tran}(\succsim)$ *is a partial order on* X*. Moreover, for any* $\trianglerighteq \in \text{Ext}(\succsim)$*, we have*

$$\succ \;\subseteq\; \text{tran}(\succsim)^{>} \subseteq \;\triangleright\,. \tag{15}$$

Proof. Suppose x tran$(\succsim)$ y tran$(\succsim)$ x for some distinct $x, y \in X$. Then, there exist finitely many (pairwise distinct) $z_0, ..., z_k, w_0, ..., w_l \in X$ such that $x = z_0 \succsim z_1 \succsim \cdots \succsim z_k = y = w_0 \succsim w_1 \succsim \cdots \succsim w_l = x$. Since $\succsim$ is antisymmetric, each $\succsim$ must hold strictly here, so we contradict acyclicity of $\succsim$. We thus conclude that tran$(\succsim)$ is antisymmetric, and hence, a partial order on X.[23]

To prove (15), note that, by definition, $\succ \subseteq \text{tran}(\succsim)$. To derive a contradiction, suppose there exist $x, y \in X$ such that $x \succ y$ but y tran$(\succsim)$ x. Then, there exist an integer $k \geq 2$ and (pairwise distinct) $z_0, ..., z_k \in X$ with $y = z_0 \succsim z_1 \succsim \cdots \succsim z_k = x$. Since $\succsim$ is antisymmetric, each $\succsim$ holds strictly, so we find $y \succ z_1 \succ \cdots \succ x \succ y$, contradicting the acyclicity of $\succsim$. This proves the first containment in (15). Next, suppose x tran$(\succsim)^{>}$ y. Then, again by antisymmetry of $\succsim$, there exist finitely many $z_0, ..., z_k \in X$ with $x = z_0 \succ z_1 \succ \cdots \succ z_k = y$. As $\trianglerighteq$ extends $\succsim$, we then have $x \triangleright z_1 \triangleright \cdots \triangleright y$, so, since $\trianglerighteq$ is transitive, we find $x \triangleright y$. This proves the second containment in (15). $\square$

Lemma 5.2. *Let* $\succsim$ *be an antisymmetric acyclic order on* $X \in$ *and* $\trianglerighteq \in \text{Ext}(\succsim)$*. Then,* tran$(\succsim)$ *is in-between* $\succsim$ *and* $\trianglerighteq$*.*

Proof. The proof is by induction on the cardinality of the set $\text{tran}(\succsim) \setminus \succsim$, say, m. Consider first the case $m = 1$. Then, $\text{tran}(\succsim) \setminus \succsim = \{(a, b)\}$ for some $a, b \in X$. In view of Lemma 5.2, $b \succ a$ cannot hold, so we have $(a, b) \in \text{Inc}(\succsim)$. Moreover, a and b are distinct (because $\succsim$ is reflexive) so we have a tran$(\succsim)^{>}$ b (because tran$(\succsim)$ is antisymmetric by Lemma 5.2). Again by Lemma 5.2, therefore, $a \triangleright b$. It follows that $\text{tran}(\succsim) = \succsim \oplus (a, b)$ and $\succsim \to \text{tran}(\succsim) \twoheadrightarrow \trianglerighteq$, which means tran$(\succsim)$ is in-between $\succsim$ and $\trianglerighteq$.

Now assume that $\succsim \to \text{tran}(\succsim) \twoheadrightarrow \trianglerighteq$ holds for every antisymmetric $\succsim \in \mathbb{A}(X)$ and $\trianglerighteq \in \text{Ext}(\succsim)$ such that $\text{tran}(\succsim) \setminus \succsim$ has $m \geq 1$ elements. To complete the induction, suppose $\succsim$ is an antisymmetric acyclic order on X with $|\text{tran}(\succsim) \setminus \succsim| = m + 1$. Pick any (a, b) in $\text{tran}(\succsim) \setminus \succsim$. By the same argument we made in the previous paragraph, we must have $(a, b) \in \text{Inc}(\succsim)$ and $a \triangleright b$. Moreover, acyclicity of $\succsim$ entails that of $\succsim_0 := \succsim \sqcup \{(a, b)\}$. (For, otherwise, there exist finitely many $z_1, ..., z_k \in X$ with $z_1 \succ \cdots \succ z_k \succ z_1$. Since $\succsim$ is acyclic, $(z_i, z_{i+1}) = (a, b)$ for some $i = 1, ..., k - 1$, and we can take $i = 1$, relabelling if necessary. But since a tran$(\succsim)$ b, there also exist finitely many $w_0, ..., w_l \in X$ with $a = w_0 \succ \cdots \succ w_l = b$. Consequently, $b = z_2 \succ \cdots \succ z_k \succ z_1 = a = w_1 \succ \cdots \succ w_l = b$, contradicting the acyclicity of $\succsim$.) Thus, $\text{tran}(\succsim_0) = \succsim \oplus (a, b)$ and $\succsim \to \succsim_0 \twoheadrightarrow \trianglerighteq$. Now notice that $\text{tran}(\succsim_0) \setminus \succsim_0$ has m many elements, so by the induction hypothesis, $\succsim_0 \to \text{tran}(\succsim_0) \twoheadrightarrow \trianglerighteq$. It follows that $\succsim \to \text{tran}(\succsim_0) \twoheadrightarrow \trianglerighteq$. Since $\text{tran}(\succsim) = \text{tran}(\succsim_0)$, we are done. $\square$

Now let μ be a positive measure μ on X, take any antisymmetric $\succsim \in \mathbb{A}(X)$, and let $\trianglerighteq \in \text{Ext}(\succsim)$. Then, by Lemma 5.2, tran$(\succsim)$ is in-between $\succsim$ and $\trianglerighteq$. As

[23]We give this argument here only for the sake of completeness. It is well-known that an antisymmetric binary relation on a finite set is acyclic if and only if its transitive closure is a partial order; see, for instance, [11, Theorem 2.23].

D^μ satisfies Axiom 1, we thus get

$$D^\mu(\succsim, \trianglerighteq) = D^\mu(\succsim, \text{tran}(\succsim)) + D^\mu(\text{tran}(\succsim), \trianglerighteq) \geq D^\mu(\succsim, \text{tran}(\succsim)).$$

This completes the proof of Theorem 5.1.

6. Embedding $(\mathbb{A}(X), D)$ in a Euclidean Space

As $(\mathbb{A}(X), D^\mu)$ is a finite metric space (for any positive measure on μ on 2^X), another natural query to consider here is if we can embed this space isometrically in a Euclidean space. There are of course well-known characterizations of finite metric spaces that are isometrically embeddable in a Euclidean space; see, for instance, [8, 24, 27]. But even without resorting to such theorems, we can show easily that $(\mathbb{A}(X), D^\mu)$ is not isometrically embeddable in a Euclidean space. In fact, endowing the set $\mathbb{P}^*(X)$ of all partial orders on X with D^μ yields a metric space which is not isometric to any subset of a Euclidean space (unless X contains only two elements). More generally:

Proposition 6.1. *Let X be a finite set with $|X| \geq 3$ and μ a positive measure on 2^X. Then, $(\mathbb{P}^*(X), D^\mu)$ cannot be isometrically embedded in any strictly convex (real) normed linear space.*

Proof. Take any distinct $a, a', b \in X$ and define $\succsim_0 := \Delta_X \sqcup \{(a,b)\}$, $\succsim_1 := \Delta_X \sqcup \{(a',b)\}$ and $\trianglerighteq := \Delta_X \sqcup \{(a,b),(a',b)\}$. These are partial orders on X, and it is plain that $\succsim \to \succsim_0 \twoheadrightarrow \trianglerighteq$ and $\succsim \to \succsim_1 \twoheadrightarrow \trianglerighteq$. Now, to derive a contradiction, let us suppose that there is an isometric embedding $\varphi : \mathbb{P}^*(X) \to E$ for some strictly convex normed linear space E (whose norm is denoted as $\|\cdot\|$). By translation, we may assume $\varphi(\Delta_X) = \mathbf{0}_E$, where $\mathbf{0}_E$ is the origin of E. As D^μ satisfies Axiom 1, and φ is an isometry, we have

$$\begin{aligned} \|\varphi(\succsim_0)\| + \|\varphi(\trianglerighteq) - \varphi(\succsim_0)\| &= D^\mu(\Delta_X, \succsim_0) + D^\mu(\succsim_0, \trianglerighteq) \\ &= D^\mu(\Delta_X, \trianglerighteq) \\ &= \|\varphi(\trianglerighteq)\|. \end{aligned}$$

Since E is strictly convex, therefore, there exists a positive real number λ_0 such that $\varphi(\succsim_0) = \frac{\lambda_0}{1+\lambda_0}\varphi(\trianglerighteq)$. Precisely the same reasoning yields also that $\varphi(\succsim_1) = \frac{\lambda_1}{1+\lambda_1}\varphi(\trianglerighteq)$ for some $\lambda_1 > 0$. Since

$$\|\varphi(\succsim_0)\| = D^\mu(\Delta_X, \succsim_0) = 2^{|X|-1}\mu(\{b\}) = D^\mu(\Delta_X, \succsim_1) = \|\varphi(\succsim_1)\|,$$

we must have $\lambda_0 = \lambda_1$, and it follows that $\varphi(\succsim_0) = \varphi(\succsim_1)$, whence $\succsim_0 = \succsim_1$, a contradiction. ☐

It follows from this result that any sort of embedding of $(\mathbb{P}^*(X), D^\mu)$, and hence of $(\mathbb{P}(X), D^\mu)$ or $(\mathbb{A}(X), D^\mu)$, in a Euclidean space must involve some distortion.[24] For instance, by a famous theorem of Bourgain [7], $(\mathbb{A}(X), D^\mu)$ can be embedded in the Hilbert space ℓ_2 with distortion at most $O(\log |\mathbb{A}(X)|)$. We shall not pursue this problem in this paper any further, however.

[24]The *distortion* of a map φ from a metric space (Z,d) into a normed linear space E is defined as the product of $\sup_{w,z\in Z} \frac{\|\varphi(w)-\varphi(z)\|}{d(w,z)}$ and $\sup_{w,z\in Z} \frac{d(w,z)}{\|\varphi(w)-\varphi(z)\|}$.

7. Future Research

In conclusion, we would like to point out some directions for future research. First, there are some natural best approximation problems that one should attack. A really interesting one, for instance, concerns finding the nearest total preorder on X to any given preorder $\succsim$ on X in terms of the metric D. This sort of a study would aim at characterizing such best complete approximations algebraically as well as algorithmically. This may be particularly useful when the incompleteness of a preference relation arises due to "missing data." Moreover, it would allow approximating various decision problems and games with incomplete preferences by more standard models. In addition, it would furnish a natural index of incompleteness, namely, the minimum D distance between $\succsim$ and its projection onto the set of all complete preorders on X.

Second, one may take up the problem of deducing consensus preferences from a given family of preferences, say, by minimizing the sum of D distances from that family. These sorts of problems are NP-hard, and studied extensively in the operations research literature in terms of the Kemeny-Snell-Bogart metric. It should be very interesting to find out the consequences of replacing d_{KSB} with D in those studies.

Finally, we should note that the majority of economic models presume infinite alternative spaces, and indeed, the most well-known models of individual decision theory, such as the expected utility model under risk and uncertainty, the model of Knightian uncertainty, time discounting models, menu preferences, etc., work with preferences that are defined on an infinite alternative space. By contrast, our work in this paper depends very much on the finiteness of X, and while it is readily applicable to experiments, individual choice theory, voting, etc., it does not play well within these settings. One may, of course, always extend the top-difference semimetric D to the case of an arbitrary X by means of the formula

$$D(\succsim, \trianglerighteq) = \sup \sum_{S \subseteq X} \left| M(S, \succsim) \triangle M(S, \trianglerighteq) \right|,$$

where sup is taken over all finite subsets of X, but this seems like a rather coarse approach. (It would not, for instance, distinguish any two quasi-linear preferences on $\mathbb{R}^2$.) Extending the approach developed here to the context of infinite alternative spaces remains as a major problem for future research.

Acknowledgement We thank professors Jose Apesteguia, Miguel Ballester, Chris Chambers and Federico Echenique, and especially to Gil Riella, for their insightful comments in the development stage of this work.

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