BEST COMPLETE APPROXIMATIONS OF PREFERENCE RELATIONS

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Abstract. We investigate the problem of approximating an incomplete preference relation $\succsim$ on a finite set by a complete preference relation. We aim to obtain this approximation in such a way that the choices on the basis of two preferences, one incomplete, the other complete, have the smallest possible discrepancy in the aggregate. To this end, we use the top-difference metric on preferences, and define a *best complete approximation* of $\succsim$ as a complete preference relation nearest to $\succsim$ relative to this metric. We prove that such an approximation must be a maximal completion of $\succsim$, and that it is, in fact, any one completion of $\succsim$ with the largest index. Finally, we use these results to provide a sufficient condition for the best complete approximation of a preference to be its canonical completion. This leads to closed-form solutions to the best approximation problem in the case of several incomplete preference relations of interest.

1. Introduction

While rationality of preferences is often captured by their transitivity, their completeness is rather about the trait of decisiveness. Incomplete preferences are thus encountered in economic models to account for a rational individual's potential indecisiveness about the comparative appeal of two or more alternatives. This may arise due to lack of information, uncertainty, perception difficulties (as in just-noticeable differences), or a fundamental inability of comparing certain objects of choice with multiple attributes. There is a substantial literature in decision theory, whose beginning is clearly marked by the seminal work of Aumann [1], which provides various methods of representing incomplete preferences, and which develops theories of choice that emanate from the maximization of them.

Even when the modeler is confident that a person's preferences over a given set of alternatives are complete, they may lack data to know how some of the alternatives compare to each other in the eyes of that person. This sort of a situation arises often in models of computational social choice. In that literature, preferences of a voter are taken as incomplete simply because the observed voting data provides only partial information about voters' preferences, thereby identifying only a part of the true preferences of a voter.[1] Finally, when the decision-making unit is, in

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[1]See, for instance, Conitzer and Sandholm [9] and Konczak and Lang [19]. This issue becomes particularly pressing in low-stakes, high-frequency voting environments, such as web search and product recommendation. For a nice survey on voting theory with such partial information, and hence with incomplete preferences, see Boutilier and Rosenschein [7].

fact, a multi-agent, say, a committee, or a family, whose preferences arise from the aggregation of several individual preferences, a prudent approach would be to model the preferences of that unit as incomplete (due to potential disagreements between its constituents). The most prominent example of this is the familiar *Pareto ordering*, which is a preference relation (for the group in question) that is obtained by intersecting the preferences of the involved agents.

Whatever is the cause of the incompleteness of a preference relation $\succsim$ (= a reflexive and transitive binary relation), a natural question is how one may best approximate it by means of a complete preference relation. This query, which does not seem to be addressed in the literature, is relevant for applications for a multitude of reasons. First, one may wish to replace the primitive, but incomplete, preferences with their best complete approximations, and derive sharper predictions about the choices of indecisive agents. Indeed, provided they are completions of the original preference relation $\succsim$, one may use these approximations to refine the set of maximal elements in a menu relative to $\succsim$. Second, in voting type of applications we mentioned above, one may replace the individual preferences with their best complete approximations, and "solve" collective choice problems by standard means accordingly. Third, as we will discuss later, one can use the distance between a preference relation and its best complete approximation to measure the extent of indecisiveness of that preference relation.

Viability of such applications rests on knowing how one may best approximate a preference relation by a complete one; this is precisely the problem we study in the present paper. To this end, we need to first agree on a way of measuring the distance between two preference relations on a given set X of alternatives. In this paper, we take X as a nonempty finite set, and view preferences as a means toward making choices from various menus (= nonempty subsets of X). Consequently, we wish to measure the distance between any two preference relations $\succsim$ and $\trianglerighteq$ on X by comparing what they entail in the way of choice (which, as usual, we take as the set of maximal elements) in each $S \subseteq X$. Thus, for each menu S, the discrepancy (= set difference) between the set $M(S, \succsim)$ of $\succsim$-maximal elements of S and the set $M(S, \trianglerighteq)$ of $\trianglerighteq$-maximal elements of S, contributes to the "distance" between $\succsim$ and $\trianglerighteq$. If we concentrate on the case where there is no *a priori* reason to distinguish between the alternatives, a natural way of quantifying this discrepancy is by looking at the cardinality of the set-difference between $M(S, \succsim)$ and $M(S, \trianglerighteq)$; we denote this cardinality as $\Delta_S(\succsim, \trianglerighteq)$. The (semi)metric we work with in this paper is thus defined by the formula:

$$D(\succsim, \trianglerighteq) := \sum_{S \subseteq X} \Delta_S(\succsim, \trianglerighteq).$$

This (semi)metric is called the *top-difference* (semi)*metric*. It was recently introduced, and examined from both axiomatic and computational viewpoints, in Nishimura and Ok [21]. It is distinguished from other metrics for preference relations on finite sets, such as the venerable Kemeny-Snell-Bogart metric d_{KSB}, by its insistence on comparing preferences from the perspective of choice. (See Section 2 for a brief comparison of D and d_{KSB} and its variants.)

Put informally, we aim to approximate preference relations by complete preference relations in a way to minimize the discrepancy of what these preferences entail for choices in potential menus. As D is primed to capture this discrepancy, we are prompted to define a *best complete approximation* (*bca*) of a preference relation $\succsim$ on X as any complete preference relation $\succsim^*$ on X that minimizes $D(\succsim, \trianglerighteq)$ over all complete preference relations $\trianglerighteq$ on X. We introduce this concept more formally in Section 3.1 and follow it with various examples to illustrate how it actually works.

It stands to reason that a best complete approximation of a preference relation $\succsim$ be a completion of that relation.[2] While the definition of D does not readily suggest this contention, it turns out that this is indeed the case. The first, and in our view, the deepest, main result of this paper (Section 4.1) is: *Every bca of a preference relation is a (maximal) completion of that relation.*

While the set of completions of a preference relation on X is much smaller than that of all complete preference relations on X, it is still very large even when X is of a modest size. This makes it rather difficult to calculate best complete approximations. To confront this problem, we introduce in Section 4.2 the notion of *index* for a preference relation, and then use our first main result to deduce the following duality theorem: *A complete preference relation on X is a bca of $\succsim$ if, and only if, it is a completion of $\succsim$ with the maximum index* (Section 4.3). As the index of a complete preference relation is given by an explicit formula – it is simply the sum of the cardinalities of the lower-contours (= down-sets) of all alternatives – this theorem, which converts our original minimization problem to a maximization problem, simplifies the computation of best complete approximations to a considerable extent.

Section 5 is devoted to some applications of our duality theorem. First, we use this result to obtain a sufficient condition for the unique *bca* of a preference relation to be its canonical completion.[3] Next, we show that several interesting partial orders satisfy this condition. In particular, we find that the best complete approximation of the containment order, which plays essential roles in the literatures on menu preferences and rankings of opportunity sets, is the cardinality ordering. Second, we show that the only *bca* of the refinement order on the set of all partitions of a finite set – this order is routinely encountered in information economics as an unanimous "preference for information" – is based solely on the number of cells that a partition possesses. Third, we prove that the only *bca* of any prefix ordering, which are encountered in syntactical models of information transmission, is its canonical completion. (For instance, the best complete approximation of the *word-order* turns out to be the *length-of-word-order*.) Finally, we show that the coordinatewise ordering on a finite two-dimensional lattice is obtained by summing up the coordinates (as in utilitarian aggregation).

We conclude the paper by pointing out a number of interesting research directions concerning best complete approximations of preference relations.

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[2]There is no reason to expect the converse be true, of course. To give an extreme example, suppose $\succsim$ cannot compare any two distinct alternatives. We then simply have no way of making choice predictions on the basis of $\succsim$. It is thus in the nature of things that the best complete approximation of $\succsim$ is the "everywhere indifferent" relation which matches the choice predictions (or lack of them thereof) of $\succsim$ perfectly. And yet, *every* complete preference relation on X is a completion of $\succsim$.

[3]The *canonical completion* of $\succsim$ is obtained by declaring the $\succsim$-maximal elements in X as indifferent, and then dropping them from X and declaring the $\succsim$-maximal elements in the remaining set as indifferent, and so on. Every member of any set obtained this way are ranked below those that belong the sets that come before it, while the $\succsim$-maximal elements in X are put on top.

2. Metric Spaces of Preference Relations

2.1. **Terminology.** We begin with reviewing the order-theoretic jargon that we adopt in this paper.

Preference Relations. Let X be a nonempty set, which we think of as a set of mutually exclusive choice alternatives. By a **preference relation** $\succsim$ on X, we simply mean a preorder (i.e., a reflexive and transitive binary relation) on X. As usual, we denote the asymmetric and symmetric parts of $\succsim$ by $\succ$ and $\sim$, respectively. For any $x, y \in X$, we say that x and y are $\succsim$**-comparable** if either $x \succsim y$ or $y \succsim x$ holds. For any nonempty subset Y of X, the **restriction** of $\succsim$ to Y is denoted by $\succsim_Y$ or $\succsim|_Y$, that is, $\succsim|_Y \equiv \succsim_Y := \succsim \cap (Y \times Y)$.

If $\succsim$ is a preference relation on X such that any two alternatives in X are $\succsim$-comparable (i.e., when $\succsim$ is a total preorder), we refer to $\succsim$ as a **complete preference relation** on X. The set of all preference relations on X, and that of all complete preference relations on X, are denoted as $\mathbb{P}(X)$ and $\mathbb{P}_{\mathrm{C}}(X)$, respectively. Any antisymmetric $\succsim$ in $\mathbb{P}(X)$ is said to be a **partial order** on X. If, in addition, $\succsim$ is total, it is called a **linear order** on X.

Completions. Let $\succsim$ be a preference relation on X. By a **completion** of $\succsim$, we mean a total preorder $\succsim^*$ on X such that $\succsim \subseteq \succsim^*$ and $\succ \subseteq \succ^*$. This definition ensures that a completion of a preference relation is faithful to that relation in terms of both indifference and strict preference. (Obviously, $\succsim$ is its own completion iff it is complete to begin with.) We refer to a completion $\succsim^*$ of $\succsim$ as a **strict completion** if either $x \succ^* y$ or $y \succ^* x$ holds for any $x, y \in X$ that are not $\succsim$-comparable. (In particular, any strict completion of a partial order is, per force, a linear order.) Finally, by a **maximal completion** of $\succsim$, we mean a completion of $\succsim$ which is not properly contained in any other completion of $\succsim$. For instance, every complete preorder on X is a completion of the equality relation on X, while the unique maximal completion of $=$ on X is the everywhere-indifferent relation $X \times X$.

Maxima. For any $\succsim \in \mathbb{P}(X)$ and nonempty $S \subseteq X$, we denote the set of all $\succsim$-maximal elements in S by $M(S, \succsim)$, that is, $x \in M(S, \succsim)$ iff $x \in S$ and $y \succ x$ holds for no $y \in S$. On the other hand, the set of all $\succsim$-maximum elements in S is denoted by $m(S, \succsim)$, that is, $m(S, \succsim) := \{x \in S : x \succsim y \text{ for all } y \in S\}$. Clearly, $m(S, \succsim) \subseteq M(S, \succsim)$ in general, and when S is finite, the latter set is nonempty. In addition, if S is finite and $M(S, \succsim)$ happens to be a singleton, we have $m(S, \succsim) = M(S, \succsim)$. Of course, this equality holds when $\succsim$ is a complete preference relation, regardless of the cardinality of the latter set.

2.2. **Metrics on $\mathbb{P}(X)$.** There are several ways one can metrize the set of all preferences on a set X. We quickly review the two standard approaches of doing this, and then introduce the metric we will adopt here instead.

The Hausdorff Metric. An important branch of mathematical economics that started with Debreu [11] takes X as a topological space, and then considers topologizing the space of preorders on X by using some hyperspace topology, such as the one that is induced by the Hausdorff metric.[4] This approach is best suited for situations where X is an infinite set, and is primed toward studying (topological)

[4]For a recent contribution to this literature, and as well as a overview of it, see Pivato [23].

problems of preference convergence as opposed to (metric) problems of approximation of preferences. As such, it is not suitable for our present purposes. One can, of course, endow a finite set X with the discrete metric, and then metrize $\mathbb{P}(X)$ by using the Hausdorff metric, but this is the same thing as endowing $\mathbb{P}(X)$ with the discrete metric which is, obviously, too coarse to be of practical use.[5]

Henceforth, X stands as an arbitrarily fixed finite set with at least two elements.

The Kemeny-Snell-Bogart Metric. The most standard approach toward metrizing $\mathbb{P}(X)$ is by means of counting the pairwise disagreements between any two preference relations. This approach was introduced in the seminal work of Kemeny and Snell [17] where it was put on axiomatic footing in the case of linear orders on X. Bogart [6] later extended this metric to the context of all partial orders on X by means of a modified system of axioms. The resulting distance function, which we call the *Kemeny-Snell-Bogart metric*, and denote by d_{KSB}, readily extends to the set of all preorders on X. Put precisely, it is defined on $\mathbb{P}(X)$ by assessing the distance between any two preorders $\succsim$ and $\trianglerighteq$ on X as the cardinality of the symmetric difference between them, that is,

$$d_{\mathrm{KSB}}(\succsim, \trianglerighteq) = |{\succsim} \triangle {\trianglerighteq}| \,.$$

In particular, the distance between two linear orders according to d_{KSB} is simply twice the total number of involved rank reversals.

While d_{KSB} is surely an interesting metric, and figure prominently in the literatures on social choice theory and operations research, it fails to capture a decision-theoretic aspect which is of great importance for economic analysis. In economics at large, a preference relation $\succsim$ is viewed mainly as a means toward making choices in the context of various menus (nonempty subsets of the grand set X), where a "choice" in a menu S on the basis of $\succsim$ is defined as a maximal element of S with respect to $\succsim$. Consequently, the more distinct the induced "choices" of two preference relations across menus are, there is reason to think of those preferences as being less similar. The following example highlights in what sense the d_{KSB} metric does not fully reflect this viewpoint.

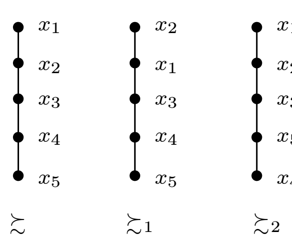

Figure 1

Example 1. Let $X := \{x_1, ..., x_5\}$, and consider the linear orders $\succsim$, $\succsim_1$ and $\succsim_2$ on X whose Hasse diagrams are depicted in Figure 1. Both $\succsim_1$ and $\succsim_2$ are obtained

[5]This approach becomes more interesting if X is endowed with a metric different than the discrete metric, but this situation falls outside the scope of this paper where we consider the choice alternatives in X as symmetric entities as in candidates in voting scenarios, or stable matchings in matching environments.

from $\succsim$ by reversing the ranks of two alternatives, namely, those of x_1 and x_2 in the case of $\succsim_1$ and those of x_4 and x_5 in the case of $\succsim_2$. Consequently, the Kemeny-Snell-Bogart metric judges the distance between $\succsim$ and $\succsim_1$ and that between $\succsim$ and $\succsim_2$ the same: $d_{\text{KSB}}(\succsim, \succsim_1) = 2 = d_{\text{KSB}}(\succsim, \succsim_2)$. But this is not supported from a choice-theoretic standpoint. Consider a person whose preferences are represented by $\succsim$. This person would never choose either x_4 or x_5 in any menu $S \subseteq X$ with the exception of $S = \{x_4, x_5\}$. Consequently, their choice behavior would differ from that of a person with preferences $\succsim_2$ in only *one* menu, namely, $\{x_4, x_5\}$. By contrast, the choice behavior entailed by $\succsim$ and $\succsim_1$ are distinct in every menu that contains x_1 and x_2. So if we observed the choices made by two people with preferences $\succsim$ and $\succsim_1$, we would see them make different choices in *eight* separate menus. From the perspective of induced choice behavior, then, it is only natural that we classify "$\succsim$ and $\succsim_1$" as being less similar than "$\succsim$ and $\succsim_2$."[6] $\square$

This example is taken from Nishimura and Ok [21] who offer several other examples to suggest that there is room for looking at alternatives to d_{KSB} and its variants, especially if we wish to distinguish between preferences on the basis of their implications for choice. It points to the fact that, at least from the perspective of implied choice behavior, the dissimilarity of two preferences depends not only on the number of rank reversals between them, but also *where* those reversals occur. In Section 3.3 we will show that another major problem with d_{KSB} is that it is too coarse to be useful for finding best complete approximations of a preference relation, the primary objective of the present paper.

The Top-Difference Metric. As an alternative to the Kemeny-Snell-Bogart metric, Nishimura and Ok [21] propose to aggregate instead the sizes of the differences in choices induced by preferences across all menus, where by a "choice induced by a preference $\succsim$ in a menu S," one means, as usual, any $\succsim$-maximal element in S.[7] On a given menu S, the dissimilarity of any two preference relations $\succsim$ and $\trianglerighteq$ on X is thus captured by comparing the sets $M(S, \succsim)$ and $M(S, \trianglerighteq)$. A particularly simple way of making this comparison is, of course, just by counting the elements in $M(S, \succsim)$ that are not in $M(S, \trianglerighteq)$, as well as those in $M(S, \trianglerighteq)$ that are not in $M(S, \succsim)$. The number of elements in the symmetric difference $M(S, \succsim) \triangle M(S, \trianglerighteq)$ tells us how different $\succsim$ and $\trianglerighteq$ are in terms of the choice behavior they entail at the menu S. We denote this number by $\Delta_S(\succsim, \trianglerighteq)$, that is,

$$\Delta_S(\succsim, \trianglerighteq) := \left| M(S, \succsim) \triangle M(S, \trianglerighteq) \right|.$$

Then, summing over all menus yields the semimetric D on $\mathbb{P}(X)$ defined as

$$D(\succsim, \trianglerighteq) := \sum_{S \subseteq X} \Delta_S(\succsim, \trianglerighteq).$$

[6] There are some well-known alternatives to d_{KSB}, such as the metrics of Blin [5], Cook and Seiford [10], and Bhattacharya and Gravel [4]. These variants are also based on the idea of counting the rank reversals between two preferences in one way or another, and also yield the same conclusion as d_{KSB} in the context of this example.

[7] For a decision-theoretic justification of defining "choices induced by $\succsim$" this way, see Eliaz and Ok [13].

As in [21], we refer to D as the **top-difference semimetric**.[8] Unlike d_{KSB}, this metric is primed to evaluate the dissimilarity of preference relations from the perspective of choice. For instance, sitting square with intuition, we have $D(\succsim, \succsim_1) = 16 > 2 = D(\succsim, \succsim_2)$ in the case of Example 1.

We note that D is a bona fide metric on the set $\mathbb{P}_{\mathrm{C}}(X)$ of all complete preference relations on X, as well as on the set of all partial orders on X. However, it serves only as a *semi*metric on $\mathbb{P}(X)$. After all, the D distance between any two preference relations on X with the same asymmetric part is zero. For instance, the distance between the equality relation on X and the everywhere-indifferent relation $X \times X$ is assessed by D as 0.

The interpretation, and hence the appeal, of the top-difference semimetric is evident at the level of its definition. For a thorough analysis of it at a foundational level, we refer the reader to [21] where a basic axiomatization for D, as well as alternative means of computing it, are provided. In what follows, we will use D to approximate preference relations with complete preference relations. This problem is described next.

3. Best Complete Approximations

3.1. **Best Complete Approximation of a Preorder.** For any preference relation $\succsim$ on X, we say that a complete preference relation $\succsim^*$ on X is a **best complete approximation** of $\succsim$ if

$$D(\succsim, \succsim^*) = \min\{D(\succsim, \trianglerighteq) : \trianglerighteq \in \mathbb{P}_{\mathrm{C}}(X)\}.$$

Given that X is finite, a best complete approximation for any $\succsim \in \mathbb{P}(X)$ always exists, but it need not be unique. We denote the set of all best complete approximations of $\succsim$ by $bca(\succsim)$, that is,

$$bca(\succsim) := \{\succsim^* \in \mathbb{P}_{\mathrm{C}}(X) : D(\succsim, \succsim^*) \leq D(\succsim, \trianglerighteq) \text{ for all } \trianglerighteq \in \mathbb{P}_{\mathrm{C}}(X)\}.$$

Clearly, bca is a nonempty set-valued map from $\mathbb{P}(X)$ onto $\mathbb{P}_{\mathrm{C}}(X)$ such that $\{\succsim\} = bca(\succsim)$ for any $\succsim \in \mathbb{P}_{\mathrm{C}}(X)$.

3.2. **Examples.** Put compactly, our purpose in this paper is to investigate the map bca. We begin with looking at some examples. Each of the following examples aim to highlight a different property of this map.

Example 2. Let $X = \{x, a, a_1, a_2\}$, and consider the partial order $\succsim$ on X whose asymmetric part is given as $a \succ a_1 \succ a_2$; the Hasse diagram of $\succsim$ is depicted in the left-most part of Figure 2.

[8] One could assess the size of $M(S, \succsim) \triangle M(S, \trianglerighteq)$ by using a measure on 2^X other than the counting measure; this generalization is pursued in [21] as well. However, in this paper we only consider the situation where the choice alternatives are symmetric, so measure the size of the difference of choice sets simply by counting the alternatives in their symmetric difference.

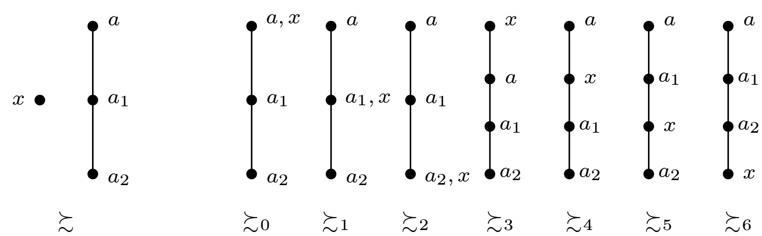

Figure 2

This partial order has exactly seven completions $\succsim_0, ..., \succsim_6$ whose Hasse diagrams are also shown in Figure 2. Here we have $D(\succsim, \succsim_0) = 3$, $D(\succsim, \succsim_1) = 5$, and $D(\succsim, \succsim_2) = 6$, and $D(\succsim, \succsim_i) = 7$ for $i = 3, ..., 6$. As we shall prove below that any best complete approximation of a preorder is a completion of that preorder, it follows that $bca(\succsim) = \{\succsim_0\}$. Incidentally, this example illustrates that two completions – in fact, even two maximal completions – of a partial order may stand at substantially varying distances from that partial order (relative to D). $\square$

Example 3. Let $X = \{x, a, a_1, a_2\}$, and consider the partial order $\succsim$ on X whose asymmetric part is given as $a \succ a_1 \succ a_2$ and $a \succ x$; the Hasse diagram of $\succsim$ is depicted in the left-most part of Figure 3.

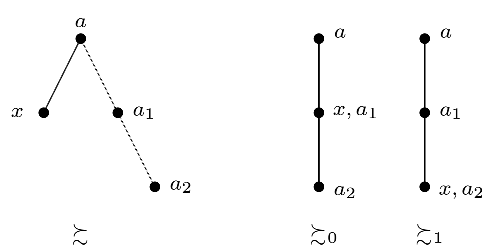

Figure 3

This partial order has exactly two maximal completions $\succsim_0$ and $\succsim_1$ whose Hasse diagrams are depicted in Figure 3. As we shall prove below that any best complete approximation of a preorder is a maximal completion of that preorder, the one that is closer to $\succsim$ is the best complete approximation of $\succsim$. As $D(\succsim, \succsim_0) = 1 < 2 = D(\succsim, \succsim_1)$, therefore, $bca(\succsim) = \{\succsim_0\}$. This example illustrates that it is essential to determine exactly which incomparable alternatives are to be declared indifferent when searching for a best complete approximation of a given preference relation. $\square$

Example 4. Let $X = \{x, y, a_1, ..., a_k\}$, where $k \geq 2$. The best complete approximation of the partial order $\succsim$ whose Hasse diagram is given in Figure 4 is the preorder $\succsim_0$ whose Hasse diagram is also given in Figure 4.

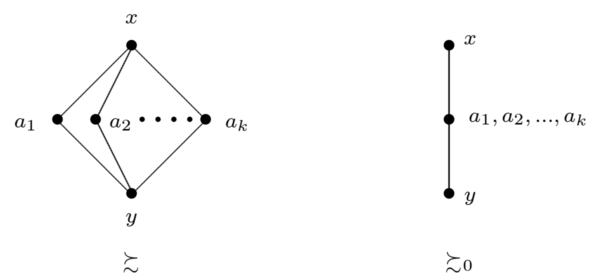

Figure 4

Here we have $D(\succsim, \succsim_0) = 0$, witnessing to the *semi*metric structure of D. □

Example 5. Let $X = \{x, a, a_1, a_2\}$, and consider the partial order $\succsim$ on X whose asymmetric part is given as $a \succ a_1$ and $a \succ a_2$; the Hasse diagram of $\succsim$ is depicted in the left-most part of Figure 5.

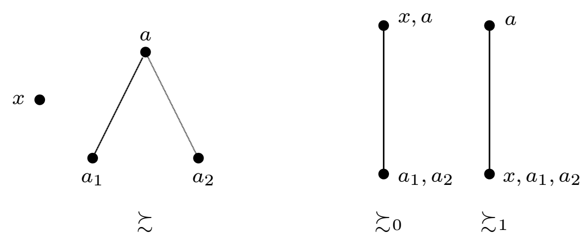

Figure 5

This partial order has two maximal completions $\succsim_0$ and $\succsim_1$ whose Hasse diagrams are also depicted in Figure 5. Consequently, by Theorem 2 (below), the one that is closer to $\succsim$ is the best complete approximation of $\succsim$. Since $D(\succsim, \succsim_0) = 4 = D(\succsim, \succsim_1)$, therefore, $bca(\succsim) = \{\succsim_0, \succsim_1\}$. This example demonstrates that a partial order may have more than one best complete approximation. □

Remark. Let $X = \{x, a_1, ..., a_9\}$, and let $\succsim$ be the preorder on X such that

$$a_1 \succ a_2 \sim a_3 \succ a_4 \sim a_5 \sim a_6 \sim a_7 \sim a_8 \sim a_9$$

with x being incomparable with any other element of X. Let $\succsim_1$ be the completion of $\succsim$ such that $x \sim a_1$, $\succsim_2$ the completion of $\succsim$ such that $x \sim a_2$, and $\succsim_3$ the completion of $\succsim$ such that $x \sim a_4$. Then, $D(\succsim, \succsim_1) = D(\succsim, \succsim_2) = D(\succsim, \succsim_3)$, and indeed we have $bca(\succsim) = \{\succsim_0, \succsim_1, \succsim_2\}$. This construction can be generalized in the obvious way to show that for every positive integer k, there is a preorder that has k many best complete approximations.

Example 6. Let $X = \{x_1, ..., x_6\}$. Consider the partial orders $\succsim_1$ and $\succsim_2$ whose Hasse diagrams are depicted in Figure 6. In the jargon of order theory, $\succsim_1$ is called the 6-**fence**, and $\succsim_2$ the 6-**crown**.

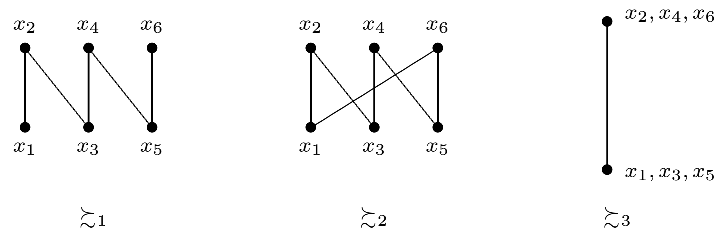

Figure 6

The only maximal completion of either $\succsim_1$ or $\succsim_2$ is the preorder $\succsim_3$ whose Hasse diagram is shown in the right-most part of Figure 6. By Theorem 2 (below), $\succsim_3$ is thus the best approximation to both $\succsim_1$ and $\succsim_2$. $\square$

3.3. **Best Complete Approximations relative to $d_{\mathbf{KSB}}$.** The best complete approximation problem is one of computing the metric-projection operator from $\mathbb{P}(X)$ onto $\mathbb{P}_{\mathrm{C}}(X)$, and it is meaningful relative to any metric on $\mathbb{P}(X)$. Given the motivation sketched above and in [21], we work in this paper only with the top-difference metric D. However, as the Kemeny-Snell-Bogart metric is the standard of the field, it should be instructive to point out what the best complete approximations of a partial order on X with respect to d_{KSB} look like. This is the content of the next observation.

Proposition 1. *Let $\succsim$ be a preference relation on X. Then,*

$$d_{\mathrm{KSB}}(\succsim, \succsim^*) = \min\{d_{\mathrm{KSB}}(\succsim, \trianglerighteq) : \trianglerighteq \in \mathbb{P}_{\mathrm{C}}(X)\}$$

for every strict completion $\succsim^*$ of $\succsim$.

Recall that a strict completion of a preference relation completes that relation in a way that strictly ranks any two previously incomparable alternatives. The upshot of Proposition 1 is that every such completion of $\succsim$ is a nearest complete preference relation to $\succsim$ relative to the Kemeny-Snell-Bogart metric.

As it is mainly of side interest for the present paper, we leave the (easy) proof of this result to the reader. We presented it here because it demonstrates that d_{KSB} is too coarse a metric to be useful for the best complete approximation problem. There may be vastly dissimilar relations in the set of all strict completions of a preference relation on X, but d_{KSB} qualifies *any* of these as a best complete approximation of that preference relation.

In particular, relative to d_{KSB}, every linear completion of a partial order is a best complete approximation of that partial order. For instance, in the case of the preference relation $\succsim$ of Example 2, $\succsim_3, \succsim_4, \succsim_5$ and $\succsim_6$ are all best complete approximations of $\succsim$ relative to d_{KSB}. These are quite distinct from each other (even by the assessment of d_{KSB}), witnessing the coarseness of d_{KSB}. This example also shows how differently D and d_{KSB} behave with respect to the best complete approximation problem.

4. Dual Characterization of Best Complete Approximations

4.1. **The Main Theorem.** A *bca* $\succsim^*$ of a preference relation $\succsim$ on X is chosen from the set of all complete preference relations on X. It seems reasonable, even desirable, that $\succsim^*$ be a completion of $\succsim$. While it is not at all designed with this goal in mind, it does deliver this property.

Theorem 2. *Every best complete approximation of a preference relation $\succsim$ on X is a maximal completion of $\succsim$.*

Determining the *bca* of a preference relation $\succsim$ on X requires minimizing $D(\succsim, \cdot)$ on the set $\mathbb{P}_{\mathrm{C}}(X)$. Foremost, Theorem 2 says that we can take as the feasible set of this problem as the set of all completions of $\succsim$. This set is much smaller than $\mathbb{P}_{\mathrm{C}}(X)$, so Theorem 2 provides substantial information about where the metric-projection of $\succsim$ is actually located within $\mathbb{P}_{\mathrm{C}}(X)$. However, for large X, the number of completions of a preorder may still be very large.[9] To counter this, Theorem 2 provides a second pointer regarding the location of *bca*($\succsim$); it says that it suffices to look only at the maximal completions of $\succsim$. We have already seen in Examples 3, 5 and 6 how useful this pointer may be.

It is not hard to prove that among the completions of a preference relation $\succsim$ on X only the maximal ones can be a best complete approximation of $\succsim$. The crux of Theorem 2 is really the fact that *bca*($\succsim$) is contained entirely within the set of completions of $\succsim$. We found proving this to be a difficult exercise. When comparing a completion $\succsim^*$ of $\succsim$ with another complete preference relation $\trianglerighteq$, it is often the case that there are many menus S for which $M(S, \succsim)$ and $M(S, \trianglerighteq)$ are closer to each other (in the sense of symmetric difference) than $M(S, \succsim)$ and $M(S, \succsim^*)$ are. Consequently, proving that the sum of the cardinalities of $M(S, \succsim) \triangle M(S, \trianglerighteq)$ exceeds that of $M(S, \succsim) \triangle M(S, \succsim^*)$, where the sums run over all menus S, requires a rather intricate combinatorial argument. This is responsible for the long proof of Theorem 2 which is presented in the Appendix.

4.2. **Index of a Preorder.** We aim to use Theorem 2 to provide an operational characterization of the *bca* map. To this end, we need to develop some order-theoretic machinery.

Let $\succsim$ be a transitive relation on X. For any $x \in X$, we denote the **down-set** (or the *principal ideal*) and the **up-set** (or the *principal filter*) of x with respect to $\succsim$ by $x^{\downarrow, \succsim}$ and $x^{\uparrow, \succsim}$, respectively. That is,

$$x^{\downarrow, \succsim} := \{y \in X : x \succsim y\} \quad \text{and} \quad x^{\uparrow, \succsim} := \{y \in X : y \succsim x\}$$

for any $x \in X$. In turn, we define the **$\succsim$-score** of any $x \in X$ as the cardinality of the family of all subsets of the down-set of x with respect to $\succsim$. We denote this number by $\mathrm{score}(x, \succsim)$, that is,

$$\mathrm{score}(x, \succsim) := 2^{|x^{\downarrow, \succsim}|}, \qquad x \in X.$$

[9]For an arbitrary finite set X, an exact formula for this number is not known. It is, however, shown by Brightwell and Winkler [8] that the problem of counting all linear completions of a partial order is #P-complete (so it is at least as hard as an NP-complete problem). It is not known if any #P-complete problem – in particular, determining the number of linear completions of a preorder – can be solved in polynomial time.

The **index** of a complete preorder $\succsim$ on X is defined as the sum of $\succsim$-scores of the elements of X. We denote this number by $\mathbb{I}(\succsim)$, that is,

$$\mathbb{I}(\succsim) := \sum_{x\in X} \mathrm{score}(x,\succsim).$$

This defines the index under the completeness hypothesis. We extend the index to the set of all preorders $\succsim$ on X as:

$$\mathbb{I}(\succsim) := \max\{\mathbb{I}(\succsim) : \succsim^{*}\in \mathbb{P}_{\mathrm{C}}(X,\succsim)\},$$

where $\mathbb{P}_{\mathrm{C}}(X,\succsim)$ stands for the family of all completions of $\succsim$.

We emphasize that $\mathbb{I}$ is an increasing map on $\mathbb{P}_{\mathrm{C}}(X)$ relative to the containment ordering. (Indeed, for any complete preorders $\succsim$ and $\trianglerighteq$ on X such that $\succsim\,\subseteq\,\trianglerighteq$, we have $x^{\downarrow,\succsim}\subseteq x^{\downarrow,\trianglerighteq}$ for every $x\in X$, whence $\mathbb{I}(\succsim)\leq\mathbb{I}(\trianglerighteq)$.) Consequently, we have the following more economic characterization of the index: For any preorder $\succsim$ on X:

$$\mathbb{I}(\succsim) := \max\{\mathbb{I}(\succsim) : \succsim^{*}\in \mathbb{P}^{*}_{\mathrm{C}}(X,\succsim)\},$$

where $\mathbb{P}^{*}_{\mathrm{C}}(X,\succsim)$ stands for the family of all *maximal* completions of $\succsim$.

To give a few immediate examples, let us assume that X has n elements. As the score of any element of X with respect to the everywhere-indifferent relation $X\times X$ is 2^n, we have $\mathbb{I}(X\times X)=n2^n$; this is the largest index any preorder on X may have. Since $X\times X$ is the only maximal completion of the equality relation on X, we also have $\mathbb{I}(=)=n2^n$. At the other extreme are linear orders. Let $\succsim$ be a linear order on X, and enumerate X as $\{x_1,...,x_n\}$ where $x_n\succ\cdots\succ x_1$. Then, clearly, $\mathrm{score}(x_i,\succsim)=2^i$ for each $i=1,...,n$. It follows that the index of any linear order $\succsim$ on X is $\sum^n 2^i$, that is, $\mathbb{I}(\succsim)=2(2^n-1)$; this is the smallest index any preorder on X may have.

These examples identify the lower and upper bounds on $\mathbb{I}$ which are worth putting on record:

$$2(2^n-1)\leq\mathbb{I}(\succsim)\leq n2^n \quad \text{for any } \succsim\in\mathbb{P}(X) \tag{1}$$

where n is the cardinality of X. We will make use of these bounds shortly.

Before looking at less trivial examples, we should explain why we are interested in the notion of the *index* for preorders.

4.3. **The Duality Theorem.** Our objective is to obtain an operational method of computing best complete approximations of a preference relation on X. To this end, we will use an alternative method of computing the top-difference semimetric D. This method was obtained in Nishimura and Ok [21] to show that D can be computed in polynomial time just like the Kemeny-Snell-Bogart metric. As a courtesy to the reader, we also provide a short proof for the following lemma in the Appendix.

Lemma 3. *For any preorders $\succsim$ and $\trianglerighteq$ on X, we have*

$$D(\succsim,\trianglerighteq)=\sum_{x\in X}\left[2^{n-|x^{\uparrow,\vartriangleright}|-1}+2^{n-|x^{\uparrow,\succ}|-1}-2^{\alpha_x(\succsim,\trianglerighteq)+1}\right] \tag{2}$$

where $\alpha_x(\succsim,\trianglerighteq)$ is the total number of $a\in X\backslash\{x\}$ such that neither $a\succ x$ nor $a\vartriangleright x$ holds.

This formula surely does not have the elegance, let alone the intuition, of the original definition of D. It is, however, exceptionally operational, especially when combined with Theorem 2. To see what we mean by this, fix an arbitrary preorder $\succsim$ on X, and recall that $\mathbb{P}_{\mathrm{C}}(X,\succsim)$ denotes the set of all completions of $\succsim$. We know from Theorem 2 that $bca(\succsim) \subseteq \mathbb{P}_{\mathrm{C}}(X,\succsim)$, so

$$bca(\succsim) = \arg\min\{D(\succsim,\succsim^*) : \succsim^* \in \mathbb{P}_{\mathrm{C}}(X,\succsim)\}.$$

In view of Lemma 3, therefore,

$$bca(\succsim) = \arg\min\left\{\sum_{x\in X} 2^{n-|x^{\uparrow,\succ^*}|-1} - \sum_{x\in X} 2^{\alpha_x(\succsim,\succsim^*)+1} : \succsim^* \in \mathbb{P}_{\mathrm{C}}(X,\succsim)\right\}.$$

But, clearly, $n - |x^{\uparrow,\succ^*}| = |x^{\downarrow,\succsim^*}|$ for every $\succsim^* \in \mathbb{P}_{\mathrm{C}}(X)$ and $x \in X$. Therefore,

$$bca(\succsim) = \arg\min\left\{\sum_{x\in X} 2^{|x^{\downarrow,\succsim^*}|-1} - \sum_{x\in X} 2^{\alpha_x(\succsim,\succsim^*)+1} : \succsim^* \in \mathbb{P}_{\mathrm{C}}(X,\succsim)\right\}.$$

Moreover, for any $\succsim^* \in \mathbb{P}_{\mathrm{C}}(X,\succsim)$,

$$\text{not } a \succ^* x \quad \text{implies} \quad \text{not } a \succ x$$

so, since $\succsim^*$ is total,

$$\{a \in X : \text{ not } a \succ x \text{ and not } a \succ^* x\} = \{a \in X : \text{ not } a \succ^* x\} = x^{\downarrow,\succsim^*}.$$

Consequently, $\alpha_x(\succsim,\succsim^*) = \left|x^{\downarrow,\succsim^*}\right| - 1$. It follows that

$$bca(\succsim) = \arg\min\left\{\sum_{x\in X} 2^{|x^{\downarrow,\succsim^*}|-1} - \sum_{x\in X} 2^{|x^{\downarrow,\succsim^*}|} : \succsim^* \in \mathbb{P}_{\mathrm{C}}(X,\succsim)\right\}.$$

Since $2^{k-1} - 2^k = -2^{k-1}$ for every nonnegative integer k, we thus find

$$bca(\succsim) = \arg\min\left\{-\sum_{x\in X} 2^{|x^{\downarrow,\succsim^*}|-1} : \succsim^* \in \mathbb{P}_{\mathrm{C}}(X,\succsim)\right\}.$$

Recalling that $\mathrm{score}(x,\succsim^*) := 2^{|x^{\downarrow,\succsim^*}|}$ for any $x \in X$, we conclude that $bca(\succsim)$ is the set of all completions of $\succsim$ with the largest index. Thus:

Theorem 4. *For any preference relation $\succsim$ on X,*

$$bca(\succsim) = \{\succsim^* \in \mathbb{P}_{\mathrm{C}}^*(X,\succsim) : \mathbb{I}(\succsim) = \mathbb{I}(\succsim^*)\}.$$

The *bca* of a preference relation is defined as the solution set of a constrained minimization problem. Theorem 4 is a *duality theorem* in the sense that it characterizes the *bca* of any preference relation as the solution set of a constrained maximization problem. To find the best complete approximations of a given preference relation $\succsim$ on X, it is evidently enough to identify those maximal completions of $\succsim$ with the largest index. While the definition of $bca(\succsim)$ makes it transparent why this concept is useful, it is not conducive to computing it. By contrast, the dual characterization of $bca(\succsim)$ given in Theorem 4 is significantly easier to compute. The next two examples aim to illustrate this point. We will provide more substantial applications of this duality approach in Section 5.

Example 7. Let $X = \{x, a, a_1, ..., a_k\}$ where $k \geq 2$, and consider the partial order $\succsim$ on X whose asymmetric part is given as $a \succ a_i$ for each $i = 1, ..., k$; the Hasse diagram of $\succsim$ is depicted in the left-most part of Figure 7.

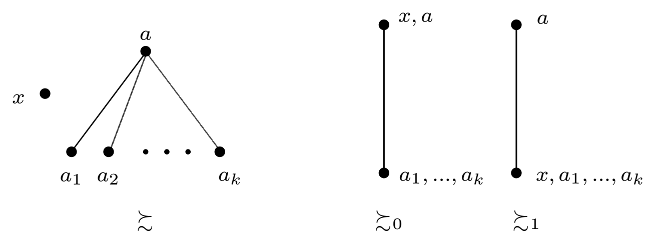

Figure 7

Clearly, in any completion of $\succsim$, we must have a above the alternatives $a_1, ..., a_k$. In turn, $a_1, ..., a_k$ can be ranked in any way among themselves (regardless of where x is ranked). But if $\succsim^*$ is a *maximal* completion of $\succsim$, we must have $a_1 \sim^* \cdots \sim^* a_k$. (Suppose this is false, and let i be the smallest number in $\{1, ..., k\}$ such that $a_j \succ^* a_i$ for some $j \in \{1, ..., k\}$, and let j be the smallest such number. Then, "moving up" a_i to the rank of a_j (that is, declaring them indifferent) while keeping all other rankings the same, yields another completion of $\succsim$ which contains $\succsim^*$ as a proper subset, contradicting the maximality of $\succsim^*$.)

It remains to determine where x is ranked in a maximal completion $\succsim^*$ of $\succsim$. The fact that $\succsim^*$ is a completion of $\succsim$ does not give any clues about this, because x is not $\succsim$-comparable with any of the other alternatives in X. But, again due to the maximality of $\succsim^*$, it must be the case that either $x \sim^* a$ or (exclusive) $x \sim^* a_1 \sim^* \cdots \sim^* a_k$. Consequently, by Theorem 2, we are sure to have $bca(\succsim) \subseteq \{\succsim_0, \succsim_1\}$, where $\succsim_0$ is the completion that corresponds to the first case and $\succsim_1$ is the completion that corresponds to the second; see Figure 7.

The indices of $\succsim_0$ and $\succsim_1$ are readily computed: $\mathbb{I}(\succsim_0) = 2(2^{k+2}) + k2^k$ and $\mathbb{I}(\succsim_1) = 2^{k+2} + (k+1)2^{k+1}$. For $k = 2$, these numbers both equal 40, so Theorem 4 says $bca(\succsim) = \{\succsim_0, \succsim_1\}$, verifying what we have already seen in Example 5. Moreover, dividing both sides of the inequality $\mathbb{I}(\succsim_0) < \mathbb{I}(\succsim_1)$ by 2^k, we see that this inequality holds iff $8+k < 4+2(k+1)$, that is, $2 < k$. By Theorem 4, therefore, $bca(\succsim) = \{\succsim_1\}$ whenever $k \geq 3$. $\square$

Remark. Example 7 illustrates that the set of maximal elements relative to a preference relation $\succsim$ may be a *proper* superset of the set of maximum elements relative to the best complete approximation of $\succsim$. Maximizing the best complete approximation of a preference relation on feasible set may thus provide a sharper prediction than maximizing that relation itself.

Example 8. Let $X := \{\alpha, x, y, a, b, c, d\}$, and consider the partial order $\succsim$ on X whose Hasse diagram is depicted in Figure 8.

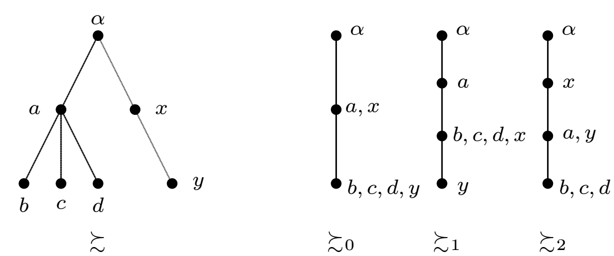

Figure 8

Reasoning as in the previous example, one can show that $\succsim$ has exactly three maximal completions $\succsim_0$, $\succsim_1$ and $\succsim_2$ whose Hasse diagrams are also presented in Figure 8. It is not readily apparent in this example which of these complete preference relations on X is closer to $\succsim$ (relative to D). This matter is readily settled by Theorem 4. Indeed, $\mathbb{I}(\succsim_0) = 2^7 + 2(2^6) + 4(2^4) = 2^7 + 192$, $\mathbb{I}(\succsim_1) = 2^7 + 2^6 + 4(2^5) + 2 = 2^7 + 194$, and $\mathbb{I}(\succsim_2) = 2^7 + 2^6 + 2(2^5) + 3(2^3) = 2^7 + 152$. We thus conclude that $bca(\succsim) = \{\succsim_1\}$. $\square$

5. Applications

In this section we compute the best complete approximations of a few well-known partial orders at a general level. In particular, we look at the all-important *containment ordering* on an arbitrary power set. We also consider the *prefix order* which is used in dynamic analysis and problems of information processing, as well as the standard *coordinatewise ordering* of $\mathbb{R}^2$, restricted to any finite grid. These applications are, in fact, special cases where the best complete approximation of a given preorder is of a particular form (which we call the canonical completion). We thus start this section by establishing a general result that gives a sufficient condition for "the" best complete approximation of a partial order to be of this form. The partial orders mentioned above are then shown to satisfy this condition. The said general result, and hence the applications of this section, rely imperatively on our duality theorem (Theorem 4).

5.1. **Canonical Completions.** Among the maximal completions of a preorder on X, an important one is its *canonical completion.* This is obtained by first identifying the maximal elements of X and then dropping those elements from X, and identifying the maximal elements of the remaining subset of X, and continuing this way inductively until the entire X is exhausted. One then declares all alternatives within each of these maximal sets indifferent, and rank the first maximal set strictly above all others, the second strictly above all others but the first one, and so on.

To define things formally, let $\succsim$ be a preorder on X. Define

$$M_1^{\succsim} := M(X, \succsim) \quad \text{and} \quad M_{i+1}^{\succsim} := M(X \backslash (M_1^{\succsim} \cup \cdots \cup M_i^{\succsim}), \succsim),\ i = 1, 2, ...$$

Let us denote by $m(\succsim)$ the largest integer m such that $M_m^{\succsim} \neq \varnothing$. Obviously, $\{M_1^{\succsim}, ..., M_{m(\succsim)}^{\succsim}\}$ is a partition of X. Consequently, the binary relation $\succsim^*$ on X is well defined by

$$x \succsim^* y \quad \text{iff} \quad (x,y) \in M_i^{\succsim} \times M_j^{\succsim} \text{ with } i \leq j.$$

It is plain that $\succsim^*$ is a maximal completion of $\succsim$; we call this total preorder the **canonical completion** of $\succsim$.

Canonical completion of a preorder is relatively easy to compute. Indeed, the definition of this completion is algorithmic to begin with. It is thus natural to ask under what sorts of conditions the canonical completion of a preference relation is indeed the best complete approximation to that preference relation. We next offer a sufficient condition for this (which is imposed on an arbitrary preference relation $\succsim$ on X).

Condition $(*)$. For any $i \in \{1, ..., m(\succsim)\}$ and nonempty proper subset S of $M_i^{\succsim}$,

$$\mathbb{I}(\succsim_Y) < 2^{|S|+|Y|}$$

where Y is the set of all $y \in X$ such that $x \succ y$ for some $x \in S$ but $x \succ y$ for no $x \in M_i \backslash S$.[10]

Admittedly, this condition does not steal one's heart at first sight. It is, however, fairly straightforward to check in specific instances. In particular, it is trivially satisfied by any linear order. It is also readily checked that it is satisfied in Examples 2, 3, 4 and 6. In the case of Example 5, it is weakly satisfied. (In that example, for $i = 1$ and $S := \{a\}$, we have $Y = \{a_1, a_2\}$, so $\mathbb{I}(\succsim_Y) = 2(2^2) = 2^{1+2} = 2^{|S|+|Y|}$.) In Example 7, the condition is again weakly satisfied for $k = 2$. However, it fails for any $k \geq 3$. To see this, note that $m(\succsim) = 2$ and $M_1^{\succsim} = \{x, a\}$ and $M_2^{\succsim} = \{a_1, ..., a_k\}$ in that example. Then, for $i = 1$ and $S = \{a\}$, we have $Y = \{a_1, ..., a_k\}$, so $\mathbb{I}(\succsim_Y) = k2^k > 2^{1+k} = 2^{|S|+|Y|}$ whenever $k > 2$. A similar analysis shows that the partial order of Example 8 fails Condition $(*)$ as well.

Out interest in Condition $(*)$ stems from the following fact, which is the final main result of the present work.

Theorem 5. *The only best complete approximation of a preference relation $\succsim$ on X that satisfies Condition $(*)$ is the canonical completion of $\succsim$.*[11]

This result is a showcase for the use of our duality theorem (Theorem 4). It seems inpenetrable with the principal definition of best complete approximations. Our proof of this result, and thus all of our subsequent applications, are based on Theorem 4.

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[10] Here Y of course depends on both i and S; we do not use a notation that makes this explicit only to simplify the statement of the condition. We also recall that $\succsim_Y$ stands for the restriction of $\succsim$ to Y (Section 2.1). In addition, it is understood here that $\mathbb{I}(\succsim_\varnothing) = 0$ so that the required inequality is trivially satisfied when $Y = \varnothing$.

[11] The proof of this result, which we present in the Appendix, will actually deliver a bit more. It will show that if Condition $(*)$ is satisfied weakly (in the sense that some (or all) of its required strict inequalities hold as equalities), then the canonical completion of the preference relation $\succsim$ belongs to $\text{bca}(\succsim)$, but it may not be only the member of $\text{bca}(\succsim)$.

5.2. **Complete Approximation of the Containment Order.** Let Z be any nonempty finite set, which we view as mutually exclusive choice prospects. A **menu-preference** is simply a preorder on the power set 2^Z. A major branch of decision theory, which was started by the seminal work of Kreps [20], is dedicated to the investigation of such preferences. In this literature, a menu is at present evaluated from the perspective of what will potentially be chosen from it at a later date. Menu preferences also figure prominently in social welfare theory where menus are interpreted as sets of (unquantifiable) opportunities (such as rights, freedoms, etc.). In this literature, a menu is valued on its own right.

Insofar as one wishes to consider menu preferences that value "flexibility" from the decision-theoretic perspective, and/or consider the elements of Z as "desirable" from the social welfare perspective, a natural condition to impose on a preference $\succsim$ on 2^Z is that it be increasing relative to the containment ordering, that is, $A \succsim B$ for every $A, B \subseteq Z$ with $A \supseteq B$. Obviously, the smallest menu-preference that respects this condition is the containment order $\supseteq$ on 2^Z itself.[12] This is a very intuitive partial order whose use is, of course, ubiquitous. (After all, every finite Boolean algebra is a power set ordered by $\supseteq$.) It is thus natural to inquire into the best way we can approximate the containment order on 2^Z by a total preorder. Our next result provides the answer.

Proposition 6. *Let Z be a nonempty finite set and $\supseteq$ the containment order on 2^Z. Then,*

$$bca(\supseteq) = \{\geq_{\mathrm{card}}\}$$

where $\geq_{\mathrm{card}}$ is the cardinality ordering on 2^Z.

Thus, the total preorder on 2^Z that is closest to the containment order on 2^Z from the perspective of menu choices is the one that ranks menus simply on the basis of the number of elements they contain. Curiously, the cardinality ordering is one of the ordering methods that has received attention in the social choice literature on preferences over sets; see, for example, Pattanaik and Xu [22] for an axiomatic characterization of this ordering, and Barberà, Bossert, and Pattanaik [3] for an excellent overview of the related literature.

In passing, we emphasize that Proposition 6 is not meant as an argument for using the cardinality ordering in practice. If there are *a priori* reasons to distinguish between the significance of the elements of Z, one would of course not pay much heed to this ordering (cf. [18]). Proposition 6 instead says that if we take the flexibility motive as the only arbiter of evaluating menus, then from the perspective of menu-choice, the one complete preference whose implied choices (in the aggregate) come closest to those that are based on that motive alone is none other than $\geq_{\mathrm{card}}$.

Remark. It may be worth noting that the conclusion of Proposition 6 is not at all what one would get if we used the Bogart-Kemeny-Snell metric instead of the top-difference metric. To wit, consider the case where $Z := \{x, y\}$. Then, there are two best complete approximations of $\supseteq$ on 2^Z with respect to d_{KSB}, neither of which is the cardinality ordering. Denoting these approximations by $\succsim_1$ and $\succsim_2$, we have $\{x, y\} \succ_1 \{x\} \succ_1 \{y\} \succ_1 \varnothing$ and $\{x, y\} \succ_2 \{y\} \succ_2 \{x\} \succ_2 \varnothing$.

Let us now turn to the proof of Proposition 6. All we need is:

[12] Ergin [14] characterizes all completions of this ordering from a decision-theoretic perspective.

Lemma 7. *Let Z be a nonempty finite set. The containment order $\supseteq$ on 2^Z satisfies Condition $(*)$.*

Proof. We have $m(\supseteq) = |Z| + 1$ and $M_i^{\supseteq} = \{S \subseteq Z : |S| = |Z| + 1 - i\}$ for each $i = 1, ..., m(\supseteq)$. Fix an arbitrary i in $\{1, ..., m(\supseteq)\}$. Since $M_1^{\supseteq}$ is a singleton (consisting only of Z), we only need consider the case $i > 1$. To simplify the notation, put $m := |Z| + 1 - i$; note that $m < |Z|$. Now take any nonempty proper subset $\mathcal{T}$ of $M_i^{\supseteq}$, and define

$$Y := \{S \subseteq Z : S \subset T \text{ for some } T \in \mathcal{T} \text{ and } S \nsubseteq T \text{ for any } T \in M_i^{\supseteq} \backslash \mathcal{T}\}.$$

We wish to show that $|Y| < 2^{|\mathcal{T}|}$. Since $\mathbb{I}(\supseteq_Y) \leq |Y| \, 2^{|Y|}$ – recall (1) – this will complete the proof of the lemma.

We first observe that for any $S \in Y$ and any $x \in Z \backslash S$, there is an m-element subset T' of Z such that $S \subset T'$ but $x \notin T'$. Indeed, for any such S and x, there is a $T \in \mathcal{T}$ with $S \subset T$. Since $\mathcal{T} \subseteq M_i^{\supseteq}$, and every element of $M_i^{\supseteq}$ has m elements, $|T| = m < |Z|$, which means $Z \backslash T \neq \varnothing$. Then, for any $y \in Z \backslash T$,

$$T' := \begin{cases} T, & \text{if } x \notin T, \\ (T \backslash \{x\}) \cup \{y\}, & \text{if } x \in T, \end{cases}$$

is an m-element subset of Z that does not contain x.

Now consider the map $f : Y \to 2^{2^Z}$ with $f(S) := \{T \subseteq Z : S \subset T \text{ and } |T| = m\}$. By the first part of the definition of Y, $f(S)$ is nonempty for every $S \in Y$. By the second part of that definition, for any $S \in Y$ and $T \in f(S)$, we have $T \in \mathcal{T}$. Thus: $f(Y) \in 2^{\mathcal{T}}$. Besides, by definition of f, we have $S \subseteq \bigcap f(S)$ for every $S \in Y$. In turn, what we have found in the previous paragraph entails that the converse containment holds as well. Thus: $S = \bigcap f(S)$ for every $S \in Y$. But then, obviously, $f(S) = f(S')$ implies $S = S'$ for any $S, S' \in Y$. This also shows that, for any $T \in \mathcal{T}$, there is no $S \in Y$ with $f(S) = \{T\}$; otherwise, $S = \bigcap f(S) = T$ while $|S| < m = |T|$. We conclude that f is a non-surjective injection from Y into $2^{\mathcal{T}}$, which means $|Y| < 2^{|\mathcal{T}|}$, completing our proof. ■

Combining Theorem 5 and Lemma 7, we see that the only best complete approximation of the containment order on the power set of a given nonempty finite set Z is its canonical completion. But it is plain that the canonical completion of the containment order on 2^Z is the cardinality ordering on 2^Z. Proposition 6 is thus proved.

5.3. **Complete Approximation of the Refinement Order.** Let Z be again a nonempty finite set, but this time let us view it as a state space in a context of uncertainty. In this context, *information* about the (unobserved) states is often modeled as partitions of Z. While its origins go about ten years earlier in the mathematics literature, this approach was pioneered in economics by Aumann [2]. Let $\mathbf{Par}(Z)$ denote the family of all partitions of Z. We refer to the elements of a partition of Z as *cells* of that partition, and denote by $\mathbf{Par}(Z, i)$ the family of all partitions of Z that have exactly i many cells, where $i = 1, ..., |Z|$.

The **refinement order** $\sqsupseteq$ is the partial order on $\mathbf{Par}(Z)$ with $\mathcal{S} \sqsupseteq \mathcal{T}$ iff for every $T \in \mathcal{T}$ there is an $S \in \mathcal{S}$ such that $S \supseteq T$. (When $\mathcal{S} \sqsupseteq \mathcal{T}$, we say that $\mathcal{T}$ is *at least as fine as* $\mathcal{S}$.) This serves as an unambiguous criterion of informativeness; if $\mathcal{S} \sqsupseteq \mathcal{T}$, then $\mathcal{T}$ is "at least as informative as" $\mathcal{S}$. In other words, the reverse of the partial order $\sqsupseteq$ can be viewed as a natural "preference for information." According to this interpretation, the most informative partition is the $\sqsupseteq$-minimum of $\mathbf{Par}(Z)$, namely, $\{\{z\} : z \in Z\}$, while the least informative partition is the $\sqsupseteq$-maximum of $\mathbf{Par}(Z)$, namely, $\{Z\}$.[13]

In this section, our goal is to determine the best complete approximation of $\sqsupseteq$. Let us begin with the end result:

Proposition 8. *Let Z be a nonempty finite set. The best complete approximation of the refinement order $\sqsupseteq$ on $\mathbf{Par}(Z)$ is the complete preorder $\sqsupseteq^*$ on $\mathbf{Par}(Z)$ with*

$$\mathcal{S} \sqsupseteq^* \mathcal{T} \quad \text{iff} \quad \mathcal{S} \text{ has at most as many cells as } \mathcal{T}.$$

We will prove this along the same lines as we proved Proposition 6 above. Indeed, it is easy to see that the order $\sqsupseteq^*$ is none other than the canonical completion of the refinement order on $\mathbf{Par}(Z)$. Consequently, Proposition 8 will follow from Theorem 5, provided we can show that $\sqsupseteq$ satisfies Condition $(*)$.

Before we do this, let us note that $\mathbf{Par}(Z)$ becomes a lattice when endowed with $\sqsupseteq$.[14] Relative to this order, the *greatest lower bound* of any nonempty subset $\mathbb{P}$ of $\mathbf{Par}(Z)$ – as usual, we denote this by $\bigwedge \mathbb{P}$ – is the partition obtained by intersecting all the cells of all the members of $\mathbb{P}$. In other words, a nonempty subset S of Z is a cell of $\bigwedge \mathbb{P}$ iff it is the largest subset of Z that fits within a single cell from each member of $\mathbb{P}$. (The *lowest upper bounds* of subsets of $\mathbf{Par}(Z)$ are a bit harder to describe, but we will not need them here.)

Observe that $m(\sqsupseteq) = |Z|$ and $M_i^{\sqsupseteq} = \mathbf{Par}(Z, i)$ for each $i = 1, ..., |Z|$. Fix an arbitrary i in $\{1, ..., |Z|\}$. Since $M_1^{\sqsupseteq}$ and $M_{|Z|}^{\sqsupseteq}$ are singletons, we only need consider the case where $|Z| > i > 1$. Now take an arbitary nonempty proper subset $\mathbb{T}$ of $M_i^{\sqsupseteq}$, and let Y stand for the set of all partitions $\mathcal{S}$ of Z such that $\mathcal{T} \sqsupset \mathcal{S}$ for some $\mathcal{T} \in \mathbb{T}$ but not $\mathcal{T} \sqsupset \mathcal{S}$ for any $\mathcal{T} \in M_i^{\sqsupseteq} \backslash \mathbb{T}$. (We assume Y is nonempty, for otherwise there is nothing to prove.) We wish to show that $|Y| < 2^{|\mathbb{T}|}$. In view of the arbitrary choice of i and $\mathbb{T}$, and because $\mathbb{I}(\sqsupseteq_Y) \leq |Y| \, 2^{|Y|}$ by (1), this will complete the proof that $\sqsupseteq$ satisfies Condition $(*)$.

Define the map $f : Y \to 2^{\mathbf{Par}(Z)}$ by

$$f(\mathcal{S}) := \{\mathcal{T} \in \mathbf{Par}(Z, i) : \mathcal{T} \sqsupset \mathcal{S}\}.$$

By the first part of the definition of Y, $f(\mathcal{S}) \neq \varnothing$ for every $\mathcal{S} \in Y$. And by the second part of that definition, for every $\mathcal{S} \in Y$ and $\mathcal{T} \in f(\mathcal{S})$, we have $\mathcal{T} \in \mathbb{T}$. Thus, the range of f is contained in $2^{\mathbb{T}}$, which means we can consider f as a map from Y into $2^{\mathbb{T}}$.

Now take any $\mathcal{S} \in Y$, and enumerate it as $\{S_1, ..., S_k\}$. Since $\mathcal{S} \in Y$, there is a partition $\mathcal{T}$ of Z with i many cells such that $\mathcal{T} \sqsupset \mathcal{S}$. Clearly, this implies $k > i \geq 2$. Moreover, by definition of f, we have $\mathcal{T} \sqsupset \mathcal{S}$ for every $\mathcal{T} \in f(S)$, that is, $\mathcal{S}$ is a $\sqsupseteq$-lower bound for $f(\mathcal{S})$. Let $\mathcal{R}$ be another $\sqsupseteq$-lower bound for $f(\mathcal{S})$. We claim that $\mathcal{S} \sqsupseteq \mathcal{R}$. To see this, take any cell R of $\mathcal{R}$. To derive a contradiction, suppose R is not contained in any of the cells of $\mathcal{S}$. Then, R must intersect at least two

[13]This way of thinking about "preference for information" is quite common in theoretical information economics. Dubra and Echenique [12], for instance, refer to any complete preference relation on $\mathbf{Par}(Z)$ that extends the reverse of $\sqsupseteq$ as *monotone*, and investigate the utility-representations of such relations.

[14]This lattice is called the *partition lattice*, and it is, in fact, universal. Indeed, a famous result of lattice theory, the *Pudlák-Tûma theorem*, says that every finite lattice can be (lattice-)embedded in a finite partition lattice.

cells of $\mathcal{S}$. Relabelling if necessary, let us suppose R intersects S_1 and S_k. Then, for $\mathcal{T} := \{S_1, ..., S_{i-1}, S_i \cup \cdots \cup S_k\}$, we have $\mathcal{T} \in \mathbf{Par}(Z, i)$ and $\mathcal{T} \sqsupset \mathcal{S}$ (whence $\mathcal{T} \in f(\mathcal{S})$) but not $\mathcal{T} \sqsupset \mathcal{R}$. This contradicts $\mathcal{R}$ being a $\sqsupseteq$-lower bound for $f(\mathcal{S})$, thereby proving our claim. We thus conclude that $\mathcal{S} = \bigwedge f(\mathcal{S})$.

In view of what we have just found, $f(\mathcal{S}) = f(\mathcal{S}')$ implies $\mathcal{S} = \mathcal{S}'$, that is, f is an injection. Besides, for any $\mathcal{T} \in \mathbb{T}$, there is no $\mathcal{S} \in Y$ with $f(\mathcal{S}) = \{\mathcal{T}\}$, for, otherwise, $\mathcal{S} = \bigwedge f(\mathcal{S}) = \mathcal{T}$, but this is impossible because $\mathcal{T} \sqsupset \mathcal{S}$. We conclude that f is a non-surjective injection from Y into $2^{\mathbb{T}}$, which means $|Y| < 2^{|\mathbb{T}|}$, as we sought.

5.4. **Complete Approximation of Prefix Orders.** A partial order $\succsim$ on a nonempty finite set X is said to be a **prefix order** if for any $x, y, z \in X$ with $x \succsim y$ and $x \succsim z$, the elements y and z are $\succsim$-comparable. Such partial orders generalize tree-orders, and are used to model "time" in models of dynamics. For, their defining condition, which is called *downward totality*, corresponds to the idea that while the "future" of a system may branch out in various ways from a given point in time, its "past" is totally ordered. In computer science, prefix orders arise also in models of information transmission, as the next example illustrates.

Example 9. Let n be any positive integer, and $\mathbb{A}$ a finite set of n elements. For an arbitrarily fixed $k \in \mathbb{N}$, we put $\Sigma^k := \mathbb{A} \cup \mathbb{A}^2 \cup \cdots \cup \mathbb{A}^k$. We may interpret the elements of X as the information encoded in n-ary form. In this context, $\mathbb{A}$ is called an *alphabet* and Σ^k is viewed as the *words* that can be obtained by means of this alphabet. The *length* of a word is simply the number of letters it contains. In turn, we think of longer words containing more information with a k-long word being the most informative one. This is captured by the partial order $\succsim$ on Σ^k defined by

$$x \succsim y \quad \text{iff} \quad y \text{ is an initial substring of } x,$$

where the latter statement means that if y is of the form $(a_1, ..., a_i)$, then either $x = y$ or x is of the form $(a_1, ..., a_i, a_{i+1}, ..., a_j)$ for some integer $j \in \{i+1, ..., k\}$. It is plain that $\succsim$, which is sometimes called the *word-order*, is a prefix order on Σ^k. $\square$

The structure of a prefix order is quite different than the containment ordering. But it turns out that their best complete approximations are obtained in the same way. This is because:

Lemma 9. *Every prefix order $\succsim$ on a finite set X satisfies Condition $(*)$.*

Proof. Fix an arbitrary i in $\{1, ..., m(\succsim)\}$, take any nonempty proper subset S of $M_i^{\succsim}$, and define

$$Y := \{y \in X : x \succ y \text{ for some } y \in S \text{ and } x \succ y \text{ for no } y \in M_i^{\succsim} \backslash S\}.$$

Let us enumerate S as $\{x_1, ..., x_k\}$, and put $S_j := \{y \in Y : x_j \succ y\}$, $j = 1, ..., k$. For each j, the downward totality property of $\succsim$ entails that the restriction of $\succsim$ (and hence of $\succsim_Y$) to S_j is total; we enumerate S_j as $\{y_{1,j}, ..., y_{n_j,j}\}$ where $y_{1,j} \succ \cdots \succ y_{n_j,j}$. Now let $\trianglerighteq$ be *any* completion of $\succsim_Y$. Then, for each $j = 1, ..., k$ and $t = 1, ..., n_j$, we have $y_t^{\uparrow,\triangleright} \supseteq \{y_{t-1}, ..., y_1, x_j\}$, so $|y_t^{\uparrow,\triangleright}| \geq t$. Given that $\trianglerighteq$

is total, and letting $n := |Y|$, this means $|y_t^{\downarrow,\trianglerighteq}| - n = -|y_t^{\uparrow,\triangleright}| \leq -t$, that is, $\text{score}(y_t, \trianglerighteq) \leq 2^n 2^{-t}$ for each such j and t. Since $Y \subseteq S_1 \cup \cdots \cup S_k$, it follows that

$$\mathbb{I}(\trianglerighteq) = \sum_{y \in Y} \text{score}(y, \trianglerighteq) \leq 2^n \sum_{j=1}^{k} \sum_{t=1}^{n_j} 2^{-t} \leq 2^n \sum_{j=1}^{k} 1 = k2^n \leq 2^{k+n}.$$

In view of the arbitrary choice of $\trianglerighteq$, we thus conclude that $\mathbb{I}(\succsim_Y) \leq 2^{|S|+|Y|}$ which was to be proved. ■

Combining Lemma 9 and Theorem 5 yields:

Proposition 10. *The only best complete approximation of a prefix order on a finite set is its canonical completion.*

Thus, in the context of Example 9, the unique best complete approximation of the word-order $\succsim$ is the complete preorder $\succsim^*$ on Σ^k defined as $x \succsim^* y$ iff the length of the word x exceeds that of y.

Remark. In the context of Example 9, the unique best complete approximation of $\precsim$ is its canonical completion as well. But $\precsim$ does not satisfy Condition $(*)$, unless $k = 1$. We thus see that Condition $(*)$ is a sufficient, but not necessary, requirement for the canonical completion of a preorder to be its unique complete approximation.

Remark. The partial orders we have considered so far in this section are structurally distinct from each other, so Propositions 6, 8 and 10 are not nested. For any nonempty finite set Z, the poset $(2^Z, \supseteq)$ is a distributive (even, Boolean) lattice, while it is well-known that $(\mathbf{Par}(Z), \sqsupseteq)$ is not modular unless $|Z| < 4$, let alone distributive. And endowing a nonempty finite set with a prefix order does not even yield a lattice in general.

5.5. **Complete Approximation of the Coordinatewise Ordering.** The most common way of ordering finite-dimensional vectors is by means of comparing them coordinate by coordinate. Restricting our attention to the two-dimensional case for simplicity, this ordering ranks a 2-vector higher than another 2-vector iff each component of the first vector is at least as large as the corresponding components of the second vector. To bring it into the realm of the present paper, we look at the restriction of this ordering to a finite (but arbitrary) grid in $\mathbb{R}^2$. Formally, take any $m \in \mathbb{N}$, and denote by z_i for the ith component of any 2-vector z. We define the **coordinatewise order** $\succsim_m$ on $\{1, ..., m\}^2$ by $x \succsim_m y$ iff $x_1 \geq y_1$ and $x_2 \geq y_2$. (If we interpret the coordinates in this setting as the utility scales of two individuals, this is none other than the familiar *Pareto ordering.*) Our question is: What is the best complete approximation of this partial order?

There are, of course, numeous ways in which one can complete the coordinatewise order (when $m > 1$). Among these, particularly interesting is the one that aggregates the coordinates additively. We denote this total preorder by $\succsim_m^+$, that is, $x \succsim_m^+ y$ iff $x_1 + x_2 \geq y_1 + y_2$. It turns out that this preference relation is the only best complete approximation of the coordinatewise order on X_m. (If, again, we look at the coordinates as utility scales, this result says that the unique bca of the Pareto ordering is obtained by means of utilitarian aggregation.)

Proposition 11. *For any positive integer m, $bca(\succsim_m) = \{\succsim_m^+\}$.*

Once again one can prove this result by first verifying that $\succsim_m$ satisfies Condition $(*)$, and then invoking Theorem 5. The required verification is not difficult, but a tad bit tedious. For brevity, we leave it to the reader.

6. Future Research

The problem of approximating incomplete preferences with complete ones is a largely unexplored area. The present paper provides only a preliminary initial investigation, and precipitates several directions for future research.

First, it seems quite desirable that we expand the set of preference relations with closed-form best complete approximations. All of the applications we reported in Section 5 have these approximations in the form of canonical completions. (Best approximations that are not canonical completions are of interest, because the maxima of such an approximation would be a proper subset of the maxima of the original (incomplete) preference relation on some menus, thereby leading to more refined predictions of choice behavior.) In particular, a concrete open problem in this regard is to determine the best complete approximations of semiorders (and even interval orders) in general, as these are not covered by our Theorem 5 and play an important role in decision theory.

Second, the best complete approximation approach leads to a natural method of quantifying how *decisive* a preference relation is. This is a fairly elusive problem. It is related to the issue of measuring the extent of incompleteness of a preference relation, but it is not quite the same problem. For instance, it is only natural that we qualify the "cannot compare anything" relation and "everywhere indifferent" relation equally decisive, because both of these relations are maximally indecisive, deeming anything choosable in any menu.[15]

The approach we outlined in this paper suggests that one may use the distance (relative to the top-difference metric) between a preference relation on X and its best complete approximation as a measure of its indecisiveness. This seems quite reasonable, but it can meaningfully compare two preference relations only when the domains of them have the same cardinality. To be able to compare the decisiveness of two preference relations that are defined on alternative spaces of varying cardinality, we need to normalize the minimum-distance computations with the largest possible minimum-distance that can be obtained in the environment. This factor is precisely the *covering radius* of $\mathbb{P}_{\mathrm{C}}(X)$ in $\mathbb{P}(X)$, that is, $\max\{D(\succsim, \succsim^*) : \succsim \in \mathbb{P}(X)$ and $\succsim^* \in \mathrm{bca}(\succsim)\})$. We do not presently know how to compute this radius for an arbitrary X.

Third, given that we work with a finite alternative space X here, it is only natural to look for algorithms to sort out the best complete approximation problem, at least with respect to some interesting classes of preference relations on X. The canonical completions can be computed algorithmically, but other than that, next to nothing is known about how to tackle the best approximation problem from a computational viewpoint.

[15] Karni and Viero [16] have recently attacked the problem of measuring the incompleteness of preferences (under risk or uncertainty) over two-outcome acts/lotteries. While very interesting, this approach does not apply to our finitistic setting (as it is based on certainty equivalences). Furthermore, it aims at measuring the extent of completeness of a preference relation, not its deciveness across menus.

Finally, we recall that the alternative spaces of most economic models are infinite, as in consumer choice theory, time preferences, or decision theory under risk and uncertainty. In these contexts, X is typically not finite, and often has itself an intrinsic metric structure. To study the best complete approximation problem in such environments, one must thus first extend the top-difference metric to the realm of preferences defined on an arbitrary metric space, which is hardly a trivial matter. With this sort of an extension at hand, or when an alternative distance function on preferences is chosen, the best complete approximation problem becomes well-defined, but solving it will require an entirely new approach. This is another wide open avenue of research which we hope to take in the future.

7. Proofs

The purpose of this section is to provide proofs for Theorem 2, Lemma 3, and Theorem 5.

7.1. **Proof of Theorem 2.** We divide the argument into two parts.

Lemma A.1. *Let $\succsim$ be a preorder on X, and $\succsim_0$ a best complete approximation of $\succsim$. Then, $\succ \subseteq \succ_0$.*

Proof. By way of contradiction, let us assume that $\succ \subseteq \succ_0$ is false. As $\succsim_0$ is total, this means

$$B := \{b \in X : a \succ b \succsim_0 a \text{ for some } a \in X\}$$

is a nonempty set. We pick any $\succsim_0$-minimal element y of B and any $x \in X$ with $x \succ y \succsim_0 x$.

Let $\succsim_1$ be the preorder on X obtained from $\succsim_0$ by pulling down the ranking of y just below x. Formally, $\succsim_1$ is the binary relation on X such that

$$\succsim_1 |_{X\backslash\{y\}} = \ \succsim_0 |_{X\backslash\{y\}}$$

and

$$\begin{cases} a \succ_1 y, & \text{if } a \succsim_0 x \\ y \succ_1 a, & \text{if } x \succ_0 a. \end{cases}$$

It is plain that $\succsim_1$ is a total preorder on X such that $x \succ_1 y$ but there is no $z \in X$ with $x \succ_1 z \succ_1 y$. Our goal is to show that $D(\succsim, \succsim_1) < D(\succsim, \succsim_0)$; this will contradict $\succsim_0$ being a best complete approximation of $\succsim$.

Consider the following sets:

$$\mathcal{A} := \left\{S \in 2^X : y \notin m(S, \succsim_0)\right\},$$

$$\mathcal{B} := \left\{S \in 2^X : \{y\} = m(S, \succsim_0)\right\},$$

and

$$\mathcal{C} := \left\{S \in 2^X : y \in m(S, \succsim_0) \neq \{y\}\right\}.$$

Obviously, $2^X = \mathcal{A} \sqcup \mathcal{B} \sqcup \mathcal{C}$. We partition $\mathcal{B}$ further into the sets

$$\mathcal{B}^1 := \{S \in \mathcal{B} : z \succsim_0 x \text{ for some } z \in S\backslash\{y\}\} \qquad \text{and} \qquad \mathcal{B}^2 := \mathcal{B}\backslash\mathcal{B}^1,$$

and $\mathcal{C}$ further into the sets

$$\mathcal{C}^1 := \{S \in \mathcal{C} : y \in M(S, \succsim)\} \qquad \text{and} \qquad \mathcal{C}^2 := \mathcal{C}\backslash\mathcal{C}^1.$$

Obviously,

$$2^X = \mathcal{A} \sqcup \mathcal{B}^1 \sqcup \mathcal{B}^2 \sqcup \mathcal{C}^1 \sqcup \mathcal{C}^2.$$

Now, if $S \in \mathcal{A}$, then the definition of $\succsim_1$ implies readily that $m(S, \succsim_0) = m(S, \succsim_1)$. On the other hand, if $S \in \mathcal{B}^2$, then $x \notin S$ and $y \succsim_0 x \succ_0 S\backslash\{y\}$, whence $m(S, \succsim_0) = \{y\} = m(S, \succsim_1)$ by definition of $\succsim_1$. Thus, for any $S \in \mathcal{A} \sqcup \mathcal{B}^2$, we have $M(S, \succsim)\triangle m(S, \succsim_0) = M(S, \succsim)\triangle m(S, \succsim_1)$, so

$$D(\succsim, \succsim_1) - D(\succsim, \succsim_0) = \sum_{S \in \mathcal{B}^1 \sqcup \mathcal{C}^1 \sqcup \mathcal{C}^2} \left(\Delta_S(\succsim, \succsim_1) - \Delta_S(\succsim, \succsim_0)\right). \tag{3}$$

We will next evaluate the sum of $\Delta_S(\succsim, \succsim_1) - \Delta_S(\succsim, \succsim_0)$ over $\mathcal{B}^1$, $\mathcal{C}^1$ and $\mathcal{C}^2$ separately.

Let $S \in \mathcal{B}^1$, and take any $a \in m(S, \succsim_1)$. By definition of $\mathcal{B}^1$, there is a $z \in S\backslash\{y\}$ with $z \succsim_0 x$. It then follows from the definition of $\succsim_1$ that $a \succ_1 y$. (In particular, $a \neq y$.) But again by definition of $\succsim_1$, there is no $w \in X$ with $x \succ_1 w \succ_1 y$. As $\succsim_1$ is total, therefore, we must have $a \succsim_1 x$, and hence, $a \succsim_0 x$.

Now if $y \succ a$, then since $x \succ a$, we get $x \succ a \succsim_0 x$, that is, $a \in B$. If, on the other hand, $b \succ a$ for some $b \in S\backslash\{y\}$, then since $a \succsim_1 b$ (and both a and b are distinct from y), we get $a \succsim_0 b$, so we again find $a \in B$. In other words, if a is not $\succsim$-maximal in S, it must belong to B, but in that case, since y was chosen as a $\succsim_0$-minimum of B, we get $a \succsim_0 y$ which means $\{y\} \neq \max(S, \succsim_0)$, contradicting $S \in \mathcal{B}$. Since a was chosen arbitrarily in $m(S, \succsim_1)$, this argument proves:

$$m(S, \succsim_1) \subseteq M(S, \succsim).$$

It follows that

$$\Delta_S(\succsim, \succsim_0) = |M(S, \succsim)\triangle\{y\}| \geq |M(S, \succsim)| - 1$$

while

$$\Delta_S(\succsim, \succsim_1) = |M(S, \succsim)\backslash m(S, \succsim_1)| \leq |M(S, \succsim)| - 1.$$

Conclusion:

$$\Delta_S(\succsim, \succsim_1) - \Delta_S(\succsim, \succsim_0) \leq 0 \qquad \text{for every } S \in \mathcal{B}^1. \tag{4}$$

Now take any $S \in \mathcal{C}^1$. In this case $y \in m(S, \succsim_0) \neq \{y\}$ (because $S \in \mathcal{C}$) so $m(S, \succsim_0) = m(S, \succsim_1) \sqcup y$. Thus, since y is $\succsim$-maximal in S (by definition of $\mathcal{C}^1$), we have

$$(M(S, \succsim)\triangle m(S, \succsim_0)) \sqcup y = M(S, \succsim)\triangle m(S, \succsim_1),$$

whence $\Delta_S(\succsim, \succsim_0) = \Delta_S(\succsim, \succsim_1) - 1$. Conclusion:

$$\sum_{S \in \mathcal{C}^1} \left(\Delta_S(\succsim, \succsim_1) - \Delta_S(\succsim, \succsim_0)\right) = \left|\mathcal{C}^1\right|. \tag{5}$$

Finally, take any $S \in \mathcal{C}^2$. In this case we again have $m(S, \succsim_0) = m(S, \succsim_1) \sqcup y$ (because $S \in \mathcal{C}$). Therefore, since now y is not $\succsim$-maximal in S (by definition of $\mathcal{C}^2$), we have

$$M(S, \succsim)\triangle m(S, \succsim_0) = (M(S, \succsim)\triangle m(S, \succsim_1)) \sqcup y,$$

whence $\Delta_S(\succsim, \succsim_0) = \Delta_S(\succsim, \succsim_1) + 1$. Conclusion:

$$\sum_{S \in \mathcal{C}^2} \left(\Delta_S(\succsim, \succsim_1) - \Delta_S(\succsim, \succsim_0)\right) = -\left|\mathcal{C}^2\right|. \tag{6}$$

Combining (3), (5), and (6), we find

$$D(\succsim, \succsim_1) - D(\succsim, \succsim_0) = \sum_{S \in \mathcal{B}^1} \left(\Delta_S(\succsim, \succsim_1) - \Delta_S(\succsim, \succsim_0)\right) + (\left|\mathcal{C}^1\right| - \left|\mathcal{C}^2\right|). \tag{7}$$

Now note that if $S \in \mathcal{C}^1$, then $x \notin S$. Moreover, in this case $y \notin M(S \sqcup x, \succsim)$ (because $x \succ y$), so $S \sqcup x \in \mathcal{C}^2$. Therefore, $S \mapsto S \sqcup x$ is an injection from $\mathcal{C}^1$ into $\mathcal{C}^2$, and hence

$$\left|\mathcal{C}^1\right| \leq \left|\mathcal{C}^2\right|. \tag{8}$$

To conclude the proof of Lemma 1, recall that $y \succsim_0 x$, so either $y \succ_0 x$ or $y \sim_0 x$. In the latter case, we have $\{x,y\} \in \mathcal{C}^2$ while $\{y\} \notin \mathcal{C}^1$, which shows that $S \mapsto S \sqcup x$ is not a surjection from $\mathcal{C}^1$ onto $\mathcal{C}^2$, whence $\left|\mathcal{C}^1\right| < \left|\mathcal{C}^2\right|$. In view of (4) and (7), therefore, we have $D(\succsim,\succsim_1) < D(\succsim,\succsim_0)$ when $y \sim_0 x$. On the other hand, if $y \succ_0 x$, we have $\{x,y\} \in \mathcal{B}^1$ and $\Delta_{\{x,y\}}(\succsim,\succsim_1) = 0 < 2 = \Delta_{\{x,y\}}(\succsim,\succsim_0)$ (because $M(\{x,y\},\succsim) = \{x\}$ while $\{y\} = m(\{x,y\},\succsim_0)$). Combining this observation with (4) yields

$$\sum_{S\in\mathcal{B}^1} (\Delta_S(\succsim,\succsim_1) - \Delta_S(\succsim,\succsim_0)) < 0,$$

and hence, in view of (8) and (7), we find $D(\succsim,\succsim_1) < D(\succsim,\succsim_0)$ when $y \succ_0 x$ as well. The proof of Lemma A.1 is now complete. ∎

Lemma A.2. *Let $\succsim$ be a preorder on X, and $\succsim_0$ a best complete approximation of $\succsim$. Then, $\sim\ \subseteq\ \sim_0$.*

Proof. By way of contradiction, let us assume that $\sim\ \subseteq\ \sim_0$ is false. As $\succsim_0$ is total, this means that there exist $x,y \in X$ such that

$$y \sim x \succ_0 y.$$

We let $\succsim_1$ stand for the preorder on X obtained from $\succsim_0$ by pulling down the ranking of x to the same level with y, and $\succsim_2$ for the preorder on X obtained from $\succsim_0$ by pushing up the ranking of y to the same level with x. Formally, $\succsim_1$ and $\succsim_2$ are the binary relations on X such that

$$\succsim_1 |_{X\setminus\{x\}} = \ \succsim_0 |_{X\setminus\{x\}} \qquad \text{and} \qquad \succsim_2 |_{X\setminus\{y\}} = \ \succsim_0 |_{X\setminus\{y\}}$$

and

$$\begin{cases} a \succsim_1 x, & \text{if } a \succsim_0 y \\ x \succ_1 a, & \text{if } y \succ_0 a \end{cases} \qquad \text{and} \qquad \begin{cases} a \succsim_2 y, & \text{if } a \succsim_0 x \\ y \succ_2 a, & \text{if } x \succ_0 a. \end{cases}$$

It is plain that $\succsim_1$ and $\succsim_2$ are total preorders on X.

Take any $S \subseteq X$. By Lemma A.1, we have $m(S,\succsim_0) \subseteq M(S,\succsim)$. The same is true for $\succsim_1$ and $\succsim_2$ as well. To see this, suppose a is not $\succsim$-maximal in S, that is, $b \succ a$ for some $b \in S$. If $b = x$, then $y \sim x \succ a$, so $y \succ_0 a$ by Lemma A.1, and hence $b = x \succ_1 a$ by definition of $\succsim_1$. If $a = x$, then $b \succ_0 x$ (Lemma A.1), so $b \succ_1 x = a$ by definition of $\succsim_1$. On the other hand, we have $b \succ_0 a$ (Lemma A.1), so $b \succ_1 a$ surely holds when both a and b are distinct from x. We conclude that a is not $\succsim_1$-maximal in S, as we claimed. Since the analogous reasoning applies to $\succsim_2$ as well, we conclude:

$$m(S,\succsim_i) \subseteq M(S,\succsim) \qquad \text{for every } S \subseteq X \text{ and } i = 0,1,2. \tag{9}$$

In what follows, our objective is to prove that

$$(D(\succsim,\succsim_1) - D(\succsim,\succsim_0)) + (D(\succsim,\succsim_2) - D(\succsim,\succsim_0)) < 0.$$

This will imply that either $D(\succsim,\succsim_1) < D(\succsim,\succsim_0)$ or $D(\succsim,\succsim_2) < D(\succsim,\succsim_0)$, and yield the desired contradiction to the hypothesis $\succsim_0 \in bca(\succsim)$. With this goal in mind, we note that (9) implies

$$\Delta_S(\succsim,\succsim_1) - \Delta_S(\succsim,\succsim_0) = (|M(S,\succsim)| - |m(S,\succsim_1)|) - (|M(S,\succsim)| - |m(S,\succsim_0)|)$$

for each $S \subseteq X$. We thus have

$$D(\succsim,\succsim_1) - D(\succsim,\succsim_0) = \sum_{S\subseteq X} (|m(S,\succsim_0)| - |m(S,\succsim_1)|)\,, \tag{10}$$

and similarly,

$$D(\succsim,\succsim_2) - D(\succsim,\succsim_0) = \sum_{S\subseteq X} (|m(S,\succsim_0)| - |m(S,\succsim_2)|)\,. \tag{11}$$

We will evaluate these sums by partitioning 2^X suitably.

We start with (10). First, we define

$$\mathcal{A} := \{S \in 2^X : x \notin S \text{ or } \{x\} = S\}.$$

Next, we partition X into the following sets (some of which may be empty):

$$X^1 := \{a \in X : a \succ_0 x\} \text{ and } X^2 := \{a \in X : a \sim_0 x\},$$

$$X^3 := \{a \in X : x \succ_0 a \succ_0 y\},$$

and

$$X^4 := \{a \in X : a \sim_0 y\} \text{ and } X^5 := \{a \in X : y \succ_0 a\}.$$

Then, we define

$$\mathcal{B}^i := \{S \in 2^X \backslash \{\varnothing\} : x \notin S \text{ and } m(S, \succsim_0) \subseteq X^i\}$$

for each $i = 1, ..., 5$, and observe that

$$2^X = \mathcal{A} \sqcup \{S \sqcup x : S \in \mathcal{B}\}$$

where $\mathcal{B} := \mathcal{B}^1 \sqcup \mathcal{B}^2 \sqcup \mathcal{B}^3 \sqcup \mathcal{B}^4 \sqcup \mathcal{B}^5$.

Now, if $S \in \mathcal{A}$, then $m(S, \succsim_0) = m(S, \succsim_1)$. On the other hand,

$$m(S \sqcup x, \succsim_0) = \begin{cases} m(S \sqcup x, \succsim_1), & \text{if } S \in \mathcal{B}^1 \sqcup \mathcal{B}^5 \\ m(S \sqcup x, \succsim_1) \sqcup x, & \text{if } S \in \mathcal{B}^2, \end{cases}$$

while

$$m(S \sqcup x, \succsim_0) = \{x\} \quad \text{and} \quad m(S \sqcup x, \succsim_1) = m(S, \succsim_0)$$

if $S \in \mathcal{B}^3$, and

$$m(S \sqcup x, \succsim_0) = \{x\} \quad \text{and} \quad m(S \sqcup x, \succsim_1) = m(S, \succsim_0) \sqcup x$$

if $S \in \mathcal{B}^4$. Using this information in (10) yields

$$\begin{aligned} D(\succsim, \succsim_1) - D(\succsim, \succsim_0) &= \left|\mathcal{B}^2\right| + \sum_{S \in \mathcal{B}^3} (1 - |m(S, \succsim_0)|) - \sum_{S \in \mathcal{B}^4} |m(S, \succsim_0)| \\ &= \left|\mathcal{B}^2\right| + \left|\mathcal{B}^3\right| - \sum_{S \in \mathcal{B}^3 \sqcup \mathcal{B}^4} |m(S, \succsim_0)| \,. \end{aligned}$$

Now note that $S \in \mathcal{B}^2 \sqcup \mathcal{B}^3$ iff $S = E \sqcup F$ for some nonempty $E \subseteq (X^2 \sqcup X^3) \backslash \{x\}$ and some (possibly empty) $F \subseteq X^4 \sqcup X^5$. It follows that

$$\left|\mathcal{B}^2\right| + \left|\mathcal{B}^3\right| = \left|\mathcal{B}^2 \sqcup \mathcal{B}^3\right| = (2^{\left|X^2\right| + \left|X^3\right| - 1} - 1) 2^{\left|X^4\right| + \left|X^5\right|},$$

whence

$$D(\succsim, \succsim_1) - D(\succsim, \succsim_0) = (2^{\left|X^2\right| + \left|X^3\right| - 1} - 1) 2^{\left|X^4\right| + \left|X^5\right|} - \sum_{S \in \mathcal{B}^3 \sqcup \mathcal{B}^4} |m(S, \succsim_0)| \,. \tag{12}$$

We now turn to evaluating (11). To this end, we define

$$\mathcal{A}' := \{S \in 2^X : y \notin S \text{ or } \{y\} = S\},$$

and

$$\mathcal{C}^i := \{S \in 2^X \backslash \{\varnothing\} : y \notin S \text{ and } m(S, \succsim_0) \subseteq X^i\}$$

for each $i = 1, ..., 5$. Clearly,

$$2^X = \mathcal{A} \sqcup \{S \sqcup y : S \in \mathcal{C}\}$$

where $\mathcal{C} := \mathcal{C}^1 \sqcup \mathcal{C}^2 \sqcup \mathcal{C}^3 \sqcup \mathcal{C}^4 \sqcup \mathcal{C}^5$.

Now, we have $m(S, \succsim_0) = m(S, \succsim_2)$ if $S \in \mathcal{A}'$, and $m(S \sqcup y, \succsim_0) = m(S \sqcup y, \succsim_2)$ if $S \in \mathcal{C}^1 \sqcup \mathcal{C}^5$, while

$$m(S \sqcup y, \succsim_2) = m(S \sqcup y, \succsim_0) \sqcup y$$

if $S \in \mathcal{C}^2$. On the other hand,

$$m(S \sqcup y, \succsim_0) = m(S, \succsim_0) \quad \text{and} \quad m(S \sqcup y, \succsim_2) = \{y\}$$

if $S \in \mathcal{C}^3$, and

$$m(S \sqcup y, \succsim_0) = m(S, \succsim_0) \sqcup y \quad \text{and} \quad m(S \sqcup y, \succsim_2) = \{y\}$$

if $S \in \mathcal{C}^4$. Using this information in (11) yields

$$\begin{aligned} D(\succsim, \succsim_2) - D(\succsim, \succsim_0) &= -\left|\mathcal{C}^2\right| + \sum_{S \in \mathcal{C}^3} (|m(S, \succsim_0)| - 1) + \sum_{S \in \mathcal{C}^4} |m(S, \succsim_0)| \\ &= -\left|\mathcal{C}^2\right| - \left|\mathcal{C}^3\right| + \sum_{S \in \mathcal{C}^3 \sqcup \mathcal{C}^4} |m(S, \succsim_0)| \,. \end{aligned}$$

Now note that $S \in \mathcal{C}^2 \sqcup \mathcal{C}^3$ iff $S = E \sqcup F$ for some nonempty $E \subseteq X^2 \sqcup X^3$ and some (possibly empty) $F \subseteq (X^4 \sqcup X^5)\backslash\{y\}$. It follows that

$$\left|\mathcal{C}^2\right| + \left|\mathcal{C}^3\right| = \left|\mathcal{C}^2 \sqcup \mathcal{C}^3\right| = (2^{|X^2|+|X^3|} - 1)2^{|X^4|+|X^5|-1},$$

whence

$$D(\succsim, \succsim_2) - D(\succsim, \succsim_0) = -(2^{|X^2|+|X^3|} - 1)2^{|X^4|+|X^5|-1} + \sum_{S \in \mathcal{C}^3 \sqcup \mathcal{C}^4} |m(S, \succsim_0)| \,. \tag{13}$$

We next observe that

$$(2^{|X^2|+|X^3|-1} - 1)2^{|X^4|+|X^5|} - (2^{|X^2|+|X^3|} - 1)2^{|X^4|+|X^5|-1} = -2^{|X^4|+|X^5|-1}.$$

As $y \in X^4 \sqcup X^5$, this number is negative, so combining (12) and (13) yields

$$(D(\succsim, \succsim_1) - D(\succsim, \succsim_0)) + (D(\succsim, \succsim_2) - D(\succsim, \succsim_0)) < \sum_{S \in \mathcal{C}^3 \sqcup \mathcal{C}^4} |m(S, \succsim_0)| - \sum_{S \in \mathcal{B}^3 \sqcup \mathcal{B}^4} |m(S, \succsim_0)| \,.$$

But if $S \in \mathcal{C}^3 \sqcup \mathcal{C}^4$, then S is nonempty and $x \succ_0 m(S, \succsim_0)$, and it follows that $x \notin S$, which means $S \in \mathcal{B}^3 \sqcup \mathcal{B}^4$. Thus, $\mathcal{C}^3 \sqcup \mathcal{C}^4 \subseteq \mathcal{B}^3 \sqcup \mathcal{B}^4$, and combining this fact with the above inequality yields

$$(D(\succsim, \succsim_1) - D(\succsim, \succsim_0)) + (D(\succsim, \succsim_2) - D(\succsim, \succsim_0)) < 0,$$

as we sought. ■

The proof of Theorem 2 is now easily completed. Indeed, by Lemmata A,1 and A.2, we already know that $\succsim_0$ is a completion of $\succsim$. It thus remains only to show that $\succsim_0$ is a maximal completion of $\succsim$. Again towards a contradiction, suppose there is a completion $\succsim_1$ of $\succsim$ that properly contains $\succsim_0$. Then, there exist $x, y \in X$ with $x \succsim_1 y$ but not $x \succsim_0 y$. As $\succsim_0$ is total, we have $y \succ_0 x$. Since $y \succ_0 x$ and $\succsim_0 \subseteq \succsim_1$, we have $y \succsim_1 x$. Thus, we have $x \sim_1 y$. In turn, since $\succsim_1$ is a completion of $\succsim$, this implies that either x and y are not $\succsim$-comparable or $x \sim y$. It follows that $m(\{x, y\}, \succsim_0) = \{y\}$ while $m(\{x, y\}, \succsim_1) = \{x, y\} = M(\{x, y\}, \succsim)$, whence

$$\Delta_{\{x,y\}}(\succsim, \succsim_1) - \Delta_{\{x,y\}}(\succsim, \succsim_0) = -1 < 0. \tag{14}$$

On the other hand, since $\succsim_0 \subseteq \succsim_1$ and $\succsim_1$ is a completion of $\succsim$, we have

$$m(S, \succsim_0) \subseteq m(S, \succsim_1) \subseteq M(S, \succsim) \quad \text{for every } S \subseteq X,$$

and hence

$$\Delta_S(\succsim, \succsim_1) - \Delta_S(\succsim, \succsim_0) \leq 0 \quad \text{for every } S \subseteq X. \tag{15}$$

It follows from (14) and (15) that $D(\succsim, \succsim_1) < D(\succsim, \succsim_0)$, which contradicts $\succsim_0$ being a best complete approximation of $\succsim$. Proof of Theorem 2 is now complete.

7.2. **Proof of Lemma 3.** By direct computation,

$$D(\succsim,\trianglerighteq)=\sum_{S\subseteq X}\triangle_S(\succsim,\trianglerighteq)=\sum_{S\subseteq X}\sum_{x\in S}\mathbf{1}_{\triangle_S(\succsim,\trianglerighteq)}(x)=\sum_{x\in X}\sum_{\substack{S\subseteq X\\ S\ni x}}\mathbf{1}_{\triangle_S(\succsim,\trianglerighteq)}(x)$$

In other words,

$$D(\succsim,\trianglerighteq)=\sum_{x\in X}\theta_x(\succsim,\trianglerighteq) \tag{16}$$

where $\theta_x(\succsim,\trianglerighteq)$ is the number of all subsets S of X such that $x\in M(S,\succsim)\triangle M(S,\trianglerighteq)$.

Let us now fix any $x\in X$, and calculate $\theta_x(\succsim,\trianglerighteq)$. To this end, let us define the following three sets:

$$A_x(\succsim,\trianglerighteq):=\{a\in X\backslash\{x\}:\ \text{not } a\succ x \text{ and not } a\rhd x\},$$

and

$$B_x(\succsim,\trianglerighteq):=\{a\in X\backslash\{x\}: a\succ x \text{ but not } a\rhd x\},$$

and

$$C_x(\succsim,\trianglerighteq):=\{a\in X\backslash\{x\}: a\rhd x \text{ but not } a\succ x\}.$$

Note that $\alpha_x(\succsim,\trianglerighteq)=|A_x(\succsim,\trianglerighteq)|$ by definition. Now, $x\in M(S,\succsim)\backslash M(S,\trianglerighteq)$ iff $S=\{x\}\sqcup K\sqcup L$ for some $K\subseteq A_x(\succsim,\trianglerighteq)$ and some *nonempty* $L\subseteq C_x(\succsim,\trianglerighteq)$. There are exactly $2^{\alpha_x(\succsim,\trianglerighteq)}(2^{|C_x(\succsim,\trianglerighteq)|}-1)$ many such sets. On the other hand, by the same logic, there are $2^{\alpha_x(\succsim,\trianglerighteq)}(2^{|B_x(\succsim,\trianglerighteq)|}-1)$ many subsets S of X such that $x\in M(S,\trianglerighteq)\backslash M(S,\succsim)$. It follows that

$$\theta_x(\succsim,\trianglerighteq)=2^{\alpha_x(\succsim,\trianglerighteq)}(2^{|B_x(\succsim,\trianglerighteq)|}+2^{|C_x(\succsim,\trianglerighteq)|}-2).$$

Next, notice that $A_x(\succsim,\trianglerighteq)\sqcup B_x(\succsim,\trianglerighteq)=\{a\in X\backslash\{x\}: \text{not } a\rhd x\}$, whence

$$\alpha_x(\succsim,\trianglerighteq)+|B_x(\succsim,\trianglerighteq)|=n-|x^{\uparrow,\rhd}|-1$$

where $n:=|X|$, and as we defined in Section 2.1, $x^{\uparrow,\rhd}$ is the principal ideal of x with respect to $\rhd$. Of course, the analogous reasoning shows that $\alpha_x(\succsim,\trianglerighteq)+|C_x(\succsim,\trianglerighteq)|=n-|x^{\uparrow,\succ}|-1$ as well. Consequently,

$$\theta_x(\succsim,\trianglerighteq)=2^{n-|x^{\uparrow,\rhd}|-1}+2^{n-|x^{\uparrow,\succ}|-1}-2^{\alpha_x(\succsim,\trianglerighteq)+1}.$$

Combining this finding with (16) yields (2).

7.3. **Proof of Theorem 5.** Define the function $\Psi:\mathbb{P}_C(X)\to[0,\infty)$ by

$$\Psi(\trianglerighteq):=\sum_{x\in X}2^{-|x^{\uparrow,\rhd}|},$$

and note that $\mathbb{I}(\trianglerighteq)=2^{|X|}\Psi(\trianglerighteq)$ for any $\trianglerighteq\in\mathbb{P}_C(X)$. In the context of the present proof, it will be more convenient to work with Ψ instead of $\mathbb{I}$.

Consider the function $f:\mathbb{N}\cup\mathbb{N}^2\cup\cdots\to[0,\infty)$ with $f(n):=n$ for any $n\in\mathbb{N}$, and

$$f(n_1,...,n_k):=n_1+\sum_{i=2}^{k}n_i2^{-n_1-\cdots-n_{i-1}}$$

for any integer $k \geq 2$ and $n_1, ..., n_k \in \mathbb{N}$. Obviously, $\Psi(X \times X) = \sum_{x\in X} 2^{-|\varnothing|} = |X| = f(|X|)$. In addition, for any $\trianglerighteq \in \mathbb{P}_C(X)$ distinct from $X \times X$, we have

$$\sum_{x\in X} 2^{-|x^{\uparrow,\triangleright}|} = \sum_{x\in M_1} 2^{-|\varnothing|} + \sum_{x\in M_2} 2^{-|M_1|} + \cdots + \sum_{x\in M_k} 2^{-|M_1|-\cdots-|M_{k-1}|}$$

where $k := m(\trianglerighteq) > 1$ and $M_i := M_i^{\trianglerighteq}$, $i = 1, ..., k$. Thus:

$$\Psi(\trianglerighteq) = f(|M_1|, ..., |M_{m(\trianglerighteq)}|) \quad \text{for every } \trianglerighteq \in \mathbb{P}_C(X). \tag{17}$$

In the foregoing argument we will make use of this formula as well as the following subadditivity property of the map f: For any $k, l \in \mathbb{N}$ with $k < l$, and any $(n_1, ..., n_l) \in \mathbb{N}^l$,

$$f(n_1, ..., n_l) = f(n_1, ..., n_k) + 2^{-n_1-\cdots-n_k} f(n_{k+1}, ..., n_l). \tag{18}$$

With these preparations at hand, we proceed to proving Theorem 5. Let $\succsim$ be a preorder on X, and denote the canonical completion of $\succsim$ by $\trianglerighteq$. Now take any completion $\succsim^*$ of $\succsim$, distinct from $\trianglerighteq$. Our objective is to show that $\succsim^*$ cannot be a maximizer of $\mathbb{I}$ over $\mathbb{P}_C(X, \succsim)$. Since $\mathbb{P}_C(X, \succsim)$ is finite, this will establish that $\arg\max\{\mathbb{I}(\succsim') : \succsim' \in \mathbb{P}_C(X, \succsim)\} = \{\trianglerighteq\}$. In turn, by Theorem 4, this gives $bca(\succsim) = \{\trianglerighteq\}$, proving Theorem 5.

To simplify the notation, we put $M_i := M_i^{\trianglerighteq}$ and $N_i := M_i^{\succsim^*}$ for every $i \in \mathbb{N}$. Since $\trianglerighteq$ and $\succsim^*$ are total, we have $M_1 \triangleright \cdots \triangleright M_{m(\trianglerighteq)}$ and $N_1 \succ^* \cdots \succ^* N_{m(\succsim^*)}$, while any two elements of M_i (resp., N_i) are indifferent relative to $\trianglerighteq$ (resp., $\succsim^*$) for any $i \in \mathbb{N}$.[16] Moreover, since $\succsim^* \neq \trianglerighteq$, the partitions $\mathcal{M} := \{M_1, ..., M_{m(\trianglerighteq)}\}$ and $\mathcal{N} := \{N_1, \cdots, N_{m(\succsim^*)}\}$ of X are distinct. Define

$$t := \min\{i \in \mathbb{N} : M_i \neq N_i\},$$

and put $M_{<t} := M_1 \cup \cdots \cup M_{t-1}$ and $N_{<t} := N_1 \cup \cdots \cup N_{t-1}$ with the understanding that $M_{<1} = \varnothing = N_{<1}$. By definition of t, we have $M_{<t} = N_{<t}$, so

$$N_t \subseteq \text{MAX}(X \backslash N_{<t}, \succsim) = \text{MAX}(X \backslash M_{<t}, \trianglerighteq) = M_t$$

where the first containment holds because $\succsim^*$ is a completion of $\succsim$, and the first equality holds because $\trianglerighteq$ is the canonical completion of $\succsim$. Since $N_{<t} \neq X$ – otherwise $\mathcal{M}$ and $\mathcal{N}$ would not be distinct – it is plain that $N_t \neq \varnothing$. As $N_t \neq M_t$, therefore, N_t is a nonempty proper subset of M_t. Since $M_{<t} = N_{<t}$, we thus have $\varnothing \neq M_t \backslash N_t \subseteq N_{t+1} \cup \cdots$, so

$$s := \min\{i \in \{t+1, ...\} : M_t \cap N_i \neq \varnothing\}$$

is well-defined. We put

$$A := N_s \backslash M_t \quad \text{and} \quad B := M_t \cap N_s.$$

Note that $A \cap B = \varnothing$, $A \cup B = N_s$ and $B \neq \varnothing$. In addition, $N_t \cap B = \varnothing$, because $B \subseteq N_s$ and $s > t$.

In the remainder of the proof, we put $a := |A|$, $b := |B|$ and $n_i := |N_i|$ for each $i \in \mathbb{N}$. The following claim is a key step in the argument.

Claim. $f(n_{t+1}, ..., n_{s-1}, a) < 2^{n_t}$ (with the understanding that the left-hand side equals $f(a)$ if $s = t + 1$).

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[16]For any transitive relation $\blacktriangleright$ on X and nonempty subsets A and B of X, by $A \blacktriangleright B$ we mean $a \blacktriangleright b$ for every $(a, b) \in A \times B$.

Proof of Claim. Let Y be the set of all $y \in X$ such that $x \succ y$ for some $x \in N_t$ and $x \succ y$ for no $x \in M_t \backslash N_t$. By construction, we have $N_{t+1} \cup \cdots \cup N_{s-1} \cup A \subseteq Y$ (with the understanding that $N_{t+1} \cup \cdots \cup N_{s-1} = \varnothing$ if $s = t+1$). Consequently, by (17),

$$f(n_{t+1}, ..., n_{s-1}, a) \leq \Psi(\succsim_Y^*).$$

But, $\Psi(\succsim_Y^*) = 2^{-|Y|}\mathbb{I}(\succsim_Y^*) \leq 2^{-|Y|}\mathbb{I}(\succsim_Y)$ (since $\succsim_Y^*$ is a completion of $\succsim_Y$), whereas $\mathbb{I}(\succsim_Y) < 2^{n_t + |Y|}$ because $\succsim$ satisfies Condition $(*)$. Combining these inequalities yields our claim. ■

We write the remainder of the proof for the case where $2 < t+1 < s < m(\succsim^*)$, but this is only for expositional purposes. The arguments for the cases where $t = 1$, or (inclusive) $t+1 = s$, or (inclusive) $s = m(\succsim^*)$, are entirely analogous (and actually have simpler expressions).

Let $\succsim'$ be the total preorder on X such that

$$N_1 \succ' \cdots \succ' N_{t-1} \succ' N_t \cup B \succ' N_{t+1} \succ' \cdots \succ' N_{s-1} \succ' A \succ' N'_{s+1} \succ' \cdots \succ' N_{m(\succsim^*)}$$

with any two elements in any one of these sets being declared indifferent. By using (17), and (18) twice,

$$\begin{aligned} \Psi(\succsim^*) &= f(n_1, ..., n_{s-1}, a+b, n_{s+1}, ..., n_m) \\ &= f(n_1, ..., n_{t-1}) + 2^{-p} f(n_t, ..., n_{s-1}, a+b) + 2^{-p-q} f(n_{s+1}, ..., n_m) \end{aligned}$$

where

$$p := n_1 + \cdots + n_{t-1}, \qquad q := n_t + \cdots + n_{s-1} + a + b \qquad \text{and} \qquad m := m(\succsim^*).$$

Likewise,

$$\begin{aligned} \Psi(\succsim') &= f(n_1, ..., n_{t-1}, n_t + b, n_{t+1}, ..., n_{s-1}, a, n_{s+1}, ..., n_m) \\ &= f(n_1, ..., n_{t-1}) + 2^{-p} f(n_t + b, n_{t+1}, ..., n_{s-1}, a) + 2^{-p-q} f(n_{s+1}, ..., n_m). \end{aligned}$$

Since $\mathbb{I}(\succsim') > \mathbb{I}(\succsim^*)$ iff $\Psi(\succsim') > \Psi(\succsim^*)$, these calculations show that $\mathbb{I}(\succsim') > \mathbb{I}(\succsim^*)$ iff

$$f(n_t + b, n_{t+1}, ..., n_{s-1}, a) > f(n_t, ..., n_{s-1}, a+b). \tag{19}$$

If we can establish this inequality, we may then conclude that $\succsim^*$ does not maximize $\mathbb{I}$ on $\mathbb{P}_C(X, \succsim)$, thereby completing the proof of Theorem 5.

To prove (19), we first use (18) to write

$$f(n_t + b, n_{t+1}, ..., n_{s-1}, a) = n_t + b + 2^{-n_t - b} f(n_{t+1}, ..., n_{s-1}, a).$$

On the other hand, with $r := n_{t+1} + \cdots + n_{s-1}$,

$$\begin{aligned} f(n_t, ..., n_{s-1}, a+b) &= n_t + 2^{-n_t} f(n_{t+1}, ..., n_{s-1}, a+b) \\ &= n_t + 2^{-n_t}(f(n_{t+1}, ..., n_{s-1}) + 2^{-r}(a+b)) \\ &= n_t + 2^{-n_t}(f(n_{t+1}, ..., n_{s-1}) + 2^{-r}a) + 2^{-n_t - r} b \\ &= n_t + 2^{-n_t} f(n_{t+1}, ..., n_{s-1}, a) + 2^{-n_t - r} b \end{aligned}$$

where we again used (18) repeatedly. Therefore, $f(n_t + b, n_{t+1}, ..., n_{s-1}, a) - f(n_t, ..., n_{s-1}, a+b)$ equals

$$\begin{aligned} (1 - 2^{-n_t - r})b - 2^{-n_t}(1 - 2^{-b}) f(n_{t+1}, ..., n_{s-1}, a) &> (1 - 2^{-n_t - r})b - 2^{-n_t}(1 - 2^{-b})2^{n_t} \\ &= (1 - 2^{-n_t - r})b - (1 - 2^{-b}) \\ &\geq \tfrac{b}{2} - (1 - 2^{-b}). \end{aligned}$$

Here we used the Claim above to get the strict inequality, while the final inequality holds because $n_t + r > 1$.

Now define the map $F : [1, \infty) \to \mathbb{R}$ by $F(x) := \frac{x}{2} - 1 + 2^{-x}$. Clearly, $F(1) = 0$ and $2^x F'(x) = 2^{x-1} - \ln 2 \geq 1 - \ln 2 > 0$ for all $x \geq 1$. It follows that $f(x) \geq 0$ for all $x \geq 1$. In particular, $\frac{b}{2} - (1 - 2^{-b}) \geq 0$ for all $b \in \mathbb{N}$. Combining this finding with the final inequality of the previous paragraph yields (19), completing our proof.

References

[1] R. Aumann, Utility theory without the completeness axiom, *Econometrica* **30** (1962), 445-462.

[2] R. Aumann, Subjectivity and correlation in randomized strategies, *J. Math Econ.* 1 (1974), 67-96.

[3] S. Barberà, W. Bossert, and P. Pattanaik, Ranking of sets of objects, in *Handbook of Utility Theory Vol. 2*, ed. by S. Barberà, Hammond, and C. Seidl, Kluwer, Boston, 2004.

[4] M. Bhattacharya and N. Gravel, Is the preference of the majority representative?, *Math. Social Sci.* **114** (2021), 87-94.

[5] J.-M. Blin, A linear assignment formulation of the multiattribute decision problem, *Rev. Francaise Automat. Informat Recherche Operationnelle* **10** (1976), 21-32.

[6] K. Bogart, Preference structures I: Distances between transitive preference relations, *J. Math. Soc.* 3 (1973), 49-67.

[7] C. Boutilier and J. Rosenschein, Incomplete information and communication in voting, in *Handbook of Computational Social Choice*, ed. by F. Brandt, V. Conitzer, U. Endriss, J. Lang, A. Procaccia, Cambridge University Press, Cambridge, 2012.

[8] G. Brightwell and P. Winkler, Counting linear completions, *Order* 8 (1991), 225-242.

[9] V. Conitzer and T. Sandholm, Vote elicitation: complexity and strategy-proofness, in *Proceedings of AAAI-02*, 2002, 392-397.

[10] W. Cook and L. Seiford, Priority ranking and consensus formation, *Man. Science* **24** (1978) 1721-1732.

[11] G. Debreu, Neighbouring economic agents. In: La Décision. Colloques internationaux du CNRS. Centre National de la Recherche Scientifique, Paris, 1969.

[12] J. Dubra and F. Echenique, Monotone preferences over information, *B.E. J. Theor. Econ.*, Article 1, 2001.

[13] K. Eliaz and Efe A. Ok, Indifference or Indecisiveness? Choice Theoretic Foundations of Incomplete Preferences, *Games Econom. Behav.* **56** (2006), 61-86.

[14] H. Ergin, Costly contemplation, *mimeo*, 2003, MIT.

[15] Y. Kannai and B. Peleg, A note on the extension of an order on a set to the power set, *J. Econ. Theory* 32 (1984), 172 175.

[16] E. Karni and M-L. Viero, Comparative completeness: Measurement, behavioral manifestations, and elicitation, *J. Econ. Behav. Organ.* 205 (2023), 423-442.

[17] J. Kemeny and J. Snell, Preference ranking: An axiomatic approach, *Mathematical Models in the Social Sciences*, Ginn, New York, 1962, pp. 9-23.

[18] M. Klemisch-Alhlert, Freedom of choice: A comparison of different rankings of opportunity sets, *Soc. Choice Welfare* 10 (1993), 189 207.

[19] K. Konczak and J. Lang, Voting procedures with incomplete preferences, in Proceedings of the 1st *Multidisciplinary Workshop on Advances in Preference Handling* (MPref 2005), 124-129.

[20] D. Kreps, A representation theorem for "preference for flexibility," *Econometrica* 47 (1979), 565-577.

[21] H. Nishimura and Efe A. Ok, A Class of dissimilarity semimetrics for preference relations, *Math. Oper. Res.* (2023), forthcoming.

[22] P. Pattanaik and Y. Xu, On ranking opportunity sets in terms of freedom of choice, *Rech. Econ. Louvain* 56 (1990), 383 390.

[23] M. Pivato, Compact spaces of continuous preferences, *mimeo*, 2023, Université Paris I.

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