Preference Structures*

Hiroki Nishimura[†] Efe A. Ok[‡]

February 24, 2024

Abstract

We model the preferences of a decision-maker by means of two binary relations. The first of these is transitive and captures comparisons that the agent finds easy or obvious. The second one describes the choices of the agent from binary choice problems, and it is assumed to be complete. Imposing two consistency conditions on these relations yields what we call a *preference structure*. The primary goal of the paper is to study the choice behavior that arises from preference structures. We show that the choice theory developed here, while much more general, embodies existence and uniqueness properties that parallel those of the classical choice theory.

JEL Classification: D11, D81.

Keywords: Preference structures, Incomplete and nontransitive preferences, behavioral economics, boundedly rational choice, top-cycles.

1 Introduction

The classical way of describing the preferences of a person on a given set X of choice prospects is to use a binary relation on X. If, according to this relation, a prospect x is ranked

*We thank David Ahn, Alfio Giarlotta, Ozgur Evren, Bart Lipman, Michael Mandler, Pietro Ortoleva, and Joel Sobel along with the participants of the seminars given at NYU, Paris School of Economics, Princeton, SMU, UC-Berkeley, UCSD, and UPenn, for their helpful comments on an earlier version of this paper. We also gratefully acknowledge that the present paper is based on work supported by the National Science Foundation under the grants NSF-1851610 and NSF-1851639.

[†]Department of Economics, University of California Riverside. Email: hiroki.nishimura@ucr.edu.

[‡]Department of Economics and the Courant Institute of Mathematical Sciences, New York University. E-mail: efe.ok@nyu.edu.

higher than another prospect y, we understand that the agent prefers having x to having y. One typically imposes some properties on this binary relation, often corresponding to a form of rationality on the part of the agent. In the most standard scenario, we posit that this relation is complete and transitive, and then describe the items that the agent finds choosable from any given feasible menu as those that maximize it.

This model is not only elegant, but it also possesses an admirable degree of predictive power. However, its explanatory power is known to be limited. In particular, there are well-known experimental demonstrations of nontransitivity of revealed preferences of individuals (Loomes and Day (2010)), and the literature offers several models that accommodate non-transitivity of preferences, including regret theory (Loomes and Sugden (1982)), nontransitive indifference and similarity (Luce (1956), Fishburn (1970), Beja and Gilboa (1992), and Rubinstein (1988)), and framing effects (Kahneman and Tversky (1979) and Salant and Rubinstein (2008)). Similarly, if we wish to model the occasional indecisiveness of a person, then the completeness hypothesis has to be dropped (as in, say, models of multi-criteria/prior decision making). Moreover, if the decision-maker unit is a group of individuals (such as a board of directors), then positing completeness and transitivity at the outset is not at all warranted. After all, the two most standard binary relations that are relevant in this case are the Pareto ordering (which is incomplete) and the majority voting rule (which is nontransitive).

Explanatory limitations of the classical approach is, at least partly, due to its presumption that all pairwise choice problems are evaluated in the same way. This is clearly unrealistic. Depending on the context, some choices may be "easy," even "trivial," for a person, while others may be "hard" enough that she may feel justifiably insecure about them. For instance, most of us would choose the sure lottery that pays \$10 over the one that pays \$5 "easily," while we may find comparing two complicated lotteries "difficult." Or, when a committee of experts unanimously advise us that alternative x is better than y, we are likely to regard deciding between x and y an "easy" problem, but if some of the experts favor x and others y, the problem may well become "hard." Similarly, comparing two social policies would be easy for a social planner when there is unanimous agreement about these policies in the society, but the choice problem may become "difficult" if some people back one policy, and the others desire the alternate.

The upshot is that the choices of a decision maker across (subjectively) "hard" choice problems may fail the strict requirements of rationality, and hence reveal a nontransitive preference relation. (This is a well-known viewpoint in the literature; see, among others, Mandler (2005) for a formal treatment, and Costa-Gomes et al. (2022) for empirical support.) By contrast, the choices across "easy" pairwise choice problems (whichever these may be for the agent) would presumably abide by transitivity. But, unless all such problems are "easy" for the agent, these yield an incomplete ordering of the choice prospects.

While realistic, this way of looking at things cannot be captured by an approach that models one's preferences by means of a single binary relation. In this paper, we instead represent the preferences of a rational decision-maker by using two binary relations on X. The first of these, denoted as $\succsim$, captures the comparisons that the agent is perfectly comfortable with. As it is unlikely that a rational agent would exhibit a cyclical choice pattern across pairwise choice problems that she can "easily" solve, we assume $\succsim$ is reflexive and transitive, but not necessarily complete. The second binary relation, denoted as $\mathbf{R}$, is revealed by the choices of the agent in pairwise choice problems. As it is generated also by "hard" choice problems, we allow $\mathbf{R}$ to be nontransitive, but, naturally, we assume it is complete.

As $\succsim$ and $\mathbf{R}$ are meant to describe the preferences of a "rational" person, they must be consistent with each other. We thus assume that both the weak and strict parts of $\mathbf{R}$ extend those of $\succsim$, respectively. That is, if the agent feels strongly that x and y are perfect substitutes for her – this is captured by the core relation $\succsim$ declaring x and y indifferent – then the revealed preference $\mathbf{R}$ maintains that x and y are indeed indifferent. Similarly, if the agent thinks x is "obviously" strictly better than y – this is captured by $\succsim$ ranking x strictly above y – then $\mathbf{R}$ reveals precisely this.

Rationality actually demands a tighter connection between $\succsim$ and $\mathbf{R}$. Suppose $x \mathbf{R} y$ and $y \succsim z$ for some alternatives x, y and z. Thus, the agent declares x superior to y (although she may not be completely confident in this judgement) while she is sure that y is better for her than z. It then seems reasonable that the agent would prefer x over z, albeit, she may be insecure about this decision (that is, $x \mathbf{R} z$ holds, but not necessarily $x \succsim z$). As the analogous reasoning applies also to the case where $x \succsim y$ and $y \mathbf{R} z$, it makes sense to require $\mathbf{R}$ be *transitive with respect to* $\succsim$, which means $x \mathbf{R} z$ holds whenever $x \mathbf{R} y \succsim z$ or $x \succsim y \mathbf{R} z$.

We call the resulting model a "preference structure." Put precisely, a *preference structure* on X is a pair of binary relations $(\succsim, \mathbf{R})$ on X such that (i) $\succsim$ is reflexive and transitive, (ii) $\mathbf{R}$ is complete, (iii) $\mathbf{R}$ is an extension of $\succsim$ (in terms of both indifference and strict preference) and (iv) $\mathbf{R}$ is transitive with respect to $\succsim$. Evidently, this model reduces to the classical one when $\succsim = \mathbf{R}$, but in general, it is considerably richer.[1]

[1]While uncommon, describing preferences through two binary relations instead of one is not new; the way

In Section 3, we provide several examples that highlight the scope of preference structures. These include the classical model as well as the models of incomplete preferences, preferences with imperfect ability of discrimination, regret preferences, and preferences completed by the recommendations of a consultant. We then prove that for any preference structure $(\succsim, \mathbf{R})$, there is a set of preorders such that $\succsim$ is realized as the intersection of this set and $\mathbf{R}$ as the union of it. Thus, the first component of any preference structure is a unanimity ordering, while its second component is a rationalizable preference in the sense of Cherepanov et al. (2013).

Our main findings are reported in Sections 4 and 5. In Section 4, we consider how one may think of an agent making her choices on the basis of her preference structure. That is, we define the set $C(S)$ of all possible choices of an economic agent from a given feasible menu S by using a preference structure. In the classical case, this is done by setting $C(S)$ as the set of all maximum elements of S with respect to the preference relation of the agent. The situation is less clear cut in the context of an arbitrarily given preference structure $(\succsim, \mathbf{R})$. What readily follows from our interpretation is that the agent would never choose an alternative x from S if there is another alternative in S that strictly dominates x in terms of the "sure" ordering $\succsim$. Thus, $C(S)$ must be contained in $\mathbf{MAX}(S, \succsim)$, the set of all maximal elements in S with respect to $\succsim$. We then posit that the "choosable" alternatives in S should "maximize" $\mathbf{R}$ on $\mathbf{MAX}(S, \succsim)$. Unfortunately, as $\mathbf{R}$ need not be transitive, there is no *a priori* reason for the existence of such maxima, even when S contains only three alternatives. This is a problem familiar from social choice theory, and it is often addressed by using an alternative notion of optima, such as the top-cycle solution, the uncovered set, the Banks set, etc. We adopt the first of these here, and set $C(S)$ as the top-cycle in $\mathbf{MAX}(S, \succsim)$ with respect to $\mathbf{R}$. This generalizes the rational choice paradigm (because, when $\succsim = \mathbf{R}$, this specification makes $C(S)$ the set of all maxima relative to $\succsim$). In addition, it captures many interesting choice frameworks, among which are the models of rational choice with incomplete preferences, some satisficing models such as choice with constant thresholds, and certain types of sequentially rational choice procedures (Section 4.5). Yet, it is not meant to be a boundedly rational choice model; it is certainly not primed to capture phenomena like the attraction effect or limited attention. It is, instead, a model that extends the scope of the standard rational choice model, and as we

our model is situated in the existing literature is explained at the end of Section 3.1. At the outset, however, we should note that we are not aware of any work that develops a choice theory on the basis of such a preference model, which is the primary focus of our work.

shall see, one that retains a considerable amount of predictive power (Section 4.6).

There are three major issues that a choice theory must address at the outset. These concern the issues of existence, uniqueness, and behavioral characterization. We discuss these in turn.

(I) *Existence of Choice.* A choice theory without having good existence properties is not appealing. This is a primary concern for any preference model that allows for nontransitive binary choice patterns; indeed, it is one of the main reasons why nontransitive preferences are seldom used in economics. In Section 4.3, we prove that the existence properties of the present choice model matches those of the rational choice model exactly. As our first main theorem, we show that under the usual compactness and continuity hypotheses, any choice correspondence that is rationalized by a preference structure is sure to be nonempty-valued.

(II) *Recoverability of the Underlying Preference Model.* This issue concerns the *uniqueness*, hence observability, of preferences from choices. The rational choice model is, trivially, on impeccable footing in this front. Provided that the choice domain is sufficiently rich, a rational choice correspondence can be rationalized by exactly one preference relation. The situation is more complicated in the case of our choice theory. Suppose C is the observed choice correspondence of a person, and assume that it is rationalized by some (unknown) preference structure $(\succsim, \mathbf{R})$. The question is if, and to what extent, we can elicit the agent's preference structure from C. This is of import, because the core preference relation of the agent is unobservable, but it is this part of the preference structure of the agent that matters most for welfare analysis (in the sense of, say, Bernheim and Rangel (2007, 2009)).

Section 5 is devoted to this issue. We first observe that the revealed preference part $\mathbf{R}$ of agent's preference structure is uniquely identified from her C. Next, we show that while there may well be a multitude of (incomplete) preference relations that would rationalize C when coupled with $\mathbf{R}$, the set of all such (core) preferences—we denote this set by $\mathbb{P}(C)$—possesses an unexpected structure. Our second main result in this paper shows that any collection of such preferences can be combined into a single, more decisive, preference relation which, when coupled with $\mathbf{R}$, rationalizes C. Consequently, there is a *most decisive* preference relation $\succsim_C$ in $\mathbb{P}(C)$. Choice theoretically, $(\succsim, \mathbf{R})$ and $(\succsim_C, \mathbf{R})$ are "equivalent." Further, $\succsim_C$ is the preorder that allows us to make unambiguous welfare comparisons most frequently, and it is perfectly observable.

In Section 5, we also give an explicit characterization of $\succsim_C$. It turns out that the strict part of this relation corresponds precisely to the Bernheim-Rangel criterion (that is, $x \succ_C y$ iff y is never chosen in a feasible set that contains x), while its symmetric part renders two alternatives indifferent iff these alternatives are behaviorally equivalent in the sense of Eliaz and Ok (2006). In sum, from any choice correspondence rationalizable by a preference structure we can recover the preference structure (i) which rationalizes that correspondence; and (ii) whose "sure" preferences exhibit the least amount of indecisiveness compatible with that choice correspondence.

(III) *Behavioral Axiomatic Characterization.* The overall approach we adopt in this paper is that of behavioral economics. We outline a theory of preferences and choice, demonstrate how this theory extends the scope of the classical theory by means of several examples, derive basic implications of this theory, and work out its existence and uniqueness properties. An alternative, complementary approach would be that of axiomatic decision theory. This would ask for the determination of the behavioral content of our choice theory in terms of a complete axiomatic system. While this approach is essential, we do not adopt it here given the already sizable length of the present exposition. A complete characterization of the present choice model in the tradition of revealed preference theory is, instead, provided in a separate, companion paper by Evren, Nishimura and Ok (2019).

2 Nomenclature

Binary Relations. By a *binary relation* on a nonempty set X, we mean any nonempty subset of $X \times X$. But, for any binary relation $\mathbf{R}$ on X, we often adopt the usual convention of writing $x \mathbf{R} y$ instead of $(x, y) \in \mathbf{R}$. For any nonempty $Y \subseteq X$, by $x \mathbf{R} Y$, we mean $x \mathbf{R} y$ for every $y \in Y$. Moreover, for any binary relations $\mathbf{R}$ and $\mathbf{S}$ on X, we simply write $x \mathbf{R} y \mathbf{S} z$ to mean $x \mathbf{R} y$ and $y \mathbf{S} z$, and so on. For any nonempty subset S of X, the *restriction* of $\mathbf{R}$ to S is defined as the binary relation on S given by $\mathbf{R}|_S := \mathbf{R} \cap (S \times S)$.

For any element x of X, the $\mathbf{R}$-*upper set* of x is defined as $x^{\uparrow,\mathbf{R}} := \{y \in X : y \mathbf{R} x\}$, and the $\mathbf{R}$-*lower set* of x is $x^{\downarrow,\mathbf{R}} := \{y \in X : x \mathbf{R} y\}$. When either $x \mathbf{R} y$ or $y \mathbf{R} x$, we say that x and y are $\mathbf{R}$-*comparable*, and put

$$\mathrm{Inc}(\mathbf{R}) := \{(x, y) \in X \times X : x \text{ and } y \text{ are not } \mathbf{R}\text{-comparable}\}.$$

If $\mathrm{Inc}(\mathbf{R}) = \varnothing$, we say that $\mathbf{R}$ is *complete* (or *total*).

The *asymmetric* (or *strict*) *part* of a binary relation $\mathbf{R}$ on X is defined as the binary relation $\mathbf{R}^>$ on X with $x \mathbf{R}^> y$ iff $x \mathbf{R} y$ and not $y \mathbf{R} x$, and the *symmetric part* of $\mathbf{R}$ is defined as $\mathbf{R}^= := \mathbf{R} \backslash \mathbf{R}^>$. The *composition* of two binary relations $\mathbf{R}$ and $\mathbf{S}$ on X is defined as $\mathbf{R} \circ \mathbf{S} :=$

$\{(x,y) \in X \times X : x \mathbf{R} z \mathbf{S} y$ for some $z \in X\}$. We say that $\mathbf{S}$ is a *subrelation* of $\mathbf{R}$, and that $\mathbf{R}$ is a *superrelation* of $\mathbf{S}$, if $\mathbf{S} \subseteq \mathbf{R}$.

We denote the diagonal of $X \times X$ by $\triangle_X$, that is, $\triangle_X := \{(x,x) : x \in X\}$. A binary relation $\mathbf{R}$ on X is said to be *reflexive* if $\triangle_X \subseteq \mathbf{R}$, *antisymmetric* if $\mathbf{R}^= \subseteq \triangle_X$, *transitive* if $\mathbf{R} \circ \mathbf{R} \subseteq \mathbf{R}$, and *quasitransitive* if $\mathbf{R}^>$ is transitive. If $\mathbf{R}$ is reflexive and transitive, we refer to it as a *preorder* on X. (Throughout the paper, generic preorders are denoted as $\succsim$ or $\trianglerighteq$, whose asymmetric parts are denoted as $\succ$ and $\triangleright$, respectively.) Finally, an antisymmetric preorder on X is said to be a *partial order* on X. If X is endowed with a prespecified partial order, we may refer to it as a *poset*.

The *transitive closure* of a binary relation $\mathbf{R}$ on X is the smallest transitive superrelation of $\mathbf{R}$; we denote this relation by $\mathrm{tran}(\mathbf{R})$. This relation always exists; we have $x \,\mathrm{tran}(\mathbf{R})\, y$ iff there exist a $k \in \mathbb{Z}_+$ and $x_0, ..., x_k \in X$ such that $x = x_0 \mathbf{R} x_1 \mathbf{R} \cdots \mathbf{R} x_k = y$. Obviously, $\mathrm{tran}(\mathbf{R})$ is a preorder on X, provided that $\mathbf{R}$ is reflexive.

Extension of Binary Relations. Let $\mathbf{R}$ be a binary relation on X. If $\mathbf{S}$ and $\mathbf{S}^>$ are subrelations of $\mathbf{R}$ and $\mathbf{R}^>$, respectively, we say that $\mathbf{R}$ is an *extension* of $\mathbf{S}$ (or that $\mathbf{R}$ *extends* $\mathbf{S}$). If $\mathbf{R}$ extends $\mathbf{S}$ and it is total, we refer to it as a *completion* of $\mathbf{S}$.

Transitivity with Respect to another Binary Relation. Our main focus in this paper is on reflexive, but not necessarily transitive, binary relations. A useful concept in the analysis of such binary relations is the notion of *transitivity with respect to a binary relation*. Put precisely, given any two binary relations $\mathbf{R}$ and $\mathbf{S}$ on X, we say that $\mathbf{R}$ is $\mathbf{S}$-*transitive* if $\mathbf{R} \circ \mathbf{S} \subseteq \mathbf{R}$ and $\mathbf{S} \circ \mathbf{R} \subseteq \mathbf{R}$, which means that either $x \mathbf{R} y \mathbf{S} z$ or $x \mathbf{S} y \mathbf{R} z$ implies $x \mathbf{R} z$ for any $x, y, z \in X$. This notion generalizes the classical concept of transitivity, for, obviously, $\mathbf{R}$ is $\mathbf{R}$-transitive iff it is transitive.

3 Preference Structures

3.1 Introduction

Let X be a nonempty set which we take as the collection of all mutually exclusive choice prospects for a decision-making unit, or an agent, who may itself be a collection of individuals (such as a board of directors, congress, or a family). This agent is entirely confident in the preferential ranking of *some* of the alternatives in X. We model these rankings by means of a binary relation $\succsim$ on X. So, when $x \succsim y$ for some $x, y \in X$, we understand that the agent is "sure" that x is better than y for her. Of course, $\succsim$ is unobservable (because we do not know when an agent is "sure" about her preferential rankings), but our interpretation mandates $\succsim$ be reflexive and transitive. However, and this is where we begin to deviate from the standard theory of rational decision-making, there is no need for $\succsim$ to be complete. The agent may well find the comparison of some alternatives "difficult," an entirely realistic phenomenon.[2]

Suppose the agent is unable to rank two alternatives x and y with respect to $\succsim$. When confronted with the problem of choosing between x and y, one will nevertheless observe her make a decision.[3] So, in the case of this choice problem, if she chooses x over y, we say that "x is revealed preferred to y," and if we have somehow witnessed that she choose x over y at some point, and y over x in some other, we say that "x is revealed indifferent to y." As such, we model *all* pairwise rankings of the individual, "easy" ones as well as the "hard" ones, by means of a binary relation $\mathbf{R}$ on X (which is observable). The very interpretation of $\mathbf{R}$ mandates it be complete. However, it is only natural that "hard choices" may not act transitively: If the agent has chosen x over y with great difficulty, and was also conflicted about her choice between y over z, but has nonetheless chosen y over z, then it may well be the case she choose z over x (again with difficulty). This not only rings true by daily introspection, but is verified by numerous experimental studies (on the nontransitivity of preferences). Moreover, if our economic agent consists of a set of individuals, then even the most standard methods of aggregating constituent preferences (such as majority voting) may result in the revelation of nontransitive rankings.

These considerations suggest that we model the "preferences" of an economic agent by means of an ordered pair $(\succsim, \mathbf{R})$ of binary relations on X such that $\succsim$ is a preorder and $\mathbf{R}$ is complete. Moreover, these relations should be consistent in the sense that $x \succsim y$ implies $x \, \mathbf{R} \, y$; this simply means that if x is "surely" at least as desirable as y for the agent, we would

[2]To wit, the agent may be employing a committee of experts, or she may be a social planner on behalf of a collection of individuals (each with her own preference relation). In either of these cases, $\succsim$ may correspond to the rankings of prospects according to the unanimity rule. When this rule applies, the comparisons are "easy," but there may be many cases in which unanimity fails.

[3]In principle, the agent may "choose" not to make a choice, but this necessitates that at least some pairwise choice problems (those that do not include the option of not choosing) to be designated as unobservable situations. As formalized later, we abstract away from such contingencies by tacitly allowing all pairwise choice situations within our framework.

observe her choose x over y.

As a matter of fact, it makes sense to ask $\mathbf{R}$ act in coherence with $\succsim$ in a way that goes beyond this property. Suppose our agent declares that $x \,\mathbf{R}\, y$ and $y \succsim z$ for some alternatives x, y and z. We interpret this as saying that the agent likes x better than y, even though she may well be somewhat insecure about this decision, while she prefers y over z in complete confidence. But then it stands to reason that the "obvious" superiority of y over z for this agent would entail that she would like x better than z, but, of course, it is possible that she may not be secure in this judgement either (that is, $x \,\mathbf{R}\, z$ holds, but not necessarily $x \succsim z$). Consequently, and since the same reasoning applies when $x \succsim y \,\mathbf{R}\, z$ as well, it makes good sense to require $\succsim$ and $\mathbf{R}$ to satisfy the following:

$$x \,\mathbf{R}\, y \succsim z \quad \text{or} \quad x \succsim y \,\mathbf{R}\, z \quad \text{implies} \quad x \,\mathbf{R}\, z$$

for all $x, y, z \in X$. Put succinctly, we posit $\mathbf{R}$ be $\succsim$-*transitive.* This property is not only reasonable, but it also brings some discipline to the model, and allows us to learn quite a bit about $\succsim$ (which is unobservable) from $\mathbf{R}$ (which is observable).

These considerations prompt the following:

Definition. An ordered pair $(\succsim, \mathbf{R})$ is a *weak preference structure* on a nonempty set X if $\succsim$ is a preorder on X and $\mathbf{R}$ is a $\succsim$-transitive and complete superrelation of $\succsim$ on X. In this context, we refer to $\succsim$ as the *core preference relation* of the structure, and to $\mathbf{R}$ as its *revealed preference relation.*

There are two special cases of the notion of a weak preference structure $(\succsim, \mathbf{R})$ that are of immediate interest. The first one of these strengthens the connection between the core and revealed preferences by requiring $\mathbf{R}$ be an *extension* of $\succsim$. This amounts to requiring $x \succ y$ imply $x \,\mathbf{R}^{>}\, y$ (in addition to $\succsim \subseteq \mathbf{R}$), that is, if x is "surely" strictly better than y for the agent, then she would never choose y over x.[4] The second one keeps the connection between $\succsim$ and $\mathbf{R}$ as is, but require $\mathbf{R}$ itself be transitive. This model embodies a lot of rationality within, but as we shall see, it still entails a choice theory distinct from the classical rational choice theory. Furthermore, many instances of this version of weak preference structures have already been considered in the literature.

[4]This requirement is natural. For, suppose $\succsim \subseteq \mathbf{R}$ holds, but $\succ \subseteq \mathbf{R}^{>}$ fails. Then, even though it is "obvious" to the agent that x is strictly better than y (that is $x \succ y$), we may have $x \,\mathbf{R}^{=}\, y$ which means that the agent could choose y over x at some point. For instance, where x is \$1000 and y is \$0, and most agents would "surely" rank the former strictly over y, we would then allow the agent reveal herself to be indifferent between \$1000 and \$0.

Definition. A weak preference structure $(\succsim, \mathbf{R})$ on a nonempty set X is said to be a *preference structure* on X, provided that $\mathbf{R}$ is a completion of $\succsim$.[5] In turn, a (weak) preference structure is said to be a *transitive* (*weak*) *preference structure* on X, if $\mathbf{R}$ is transitive.

In the next section we will highlight the relation between weak preference structures and some similar constructs of decision theory, and present several concrete examples.

Relation to the Literature. Modeling individual preferences by means of two binary relations, one incomplete and the other complete, is not new in decision theory. Especially in the literature on decision making under uncertainty, this method is employed by a number of studies. For example, in the Anscombe-Aumann framework, Gilboa et al. (2010) have used two binary relations, the first being a preorder (à la Bewley (1986)) and the second a complete preorder (à la Gilboa and Schmeidler (1989)). In the jargon introduced above, this model is a weak transitive preference structure.

There are only few studies that employ two binary relations to model individual preferences in a general framework. Both Mandler (2005) and Danan (2008) suggest distinguishing between one's core preferences – Mandler refers to these as *psychological preferences*, and Danan as *cognitive preferences* – from her revealed preferences. Mandler's model, whose outcome space is an open subset of $\mathbb{R}^n_+$, is, essentially, a special type of preference structure – see Example 3.3 below – but Mandler's emphasis is on sequential, nontransitive choice that is nevertheless consistent with core preferences. By contrast, Danan's model is of the form $(\succsim, \mathbf{R})$, where $\succsim$ and $\mathbf{R}$ are binary relations on (a topological space) X such that both $\succsim$ and $\mathbf{R}$ are complete, and $\succ \subseteq \mathbf{R}^{>}$. Danan uses this model to suggest a method of understanding when an individual who has been observed to choose x over y is, in fact, indifferent between x and y. Notably, his model is a preference structure iff $\succsim = \mathbf{R}$, that is, in the intersection of our model and Danan's lies only the classical model of preferences.

The two models of preferences that are closest to the one we consider here are those of Giarlotta and Greco (2013) and Giarlotta and Watson (2020). Both of these papers work with a weak preference structure $(\succsim, \mathbf{R})$ on X. Giarlotta and Greco (2013) impose the following additional requirement on this model: For any two alternatives x and y, either $x \succsim y$ or $y \; \mathbf{R} \; x$. This model, which is called the *necessary and possible preference* on X, declares any two alternatives that are "hard" to compare as revealed indifferent. As such, it appears rather

[5] In what follows, when we wish to emphasize that a preference structure is not weak, we will refer to it as a *proper* preference structure.

restrictive to serve as a general model of individual preferences (but it has been useful in multi-criteria decision analysis; see Giarlotta (2019) for a survey on this matter.) On the other hand, Giarlotta and Watson (2020) consider, instead, imposing the (mutually exclusive) requirements $\succ \subseteq \mathbf{R}^{>}$ or $\mathbf{R}^{>} \subseteq \succ$ on $(\succsim, \mathbf{R})$. In their jargon, the first of these leads to *complete monotonic bi-preferences* and the second to *complete comonotonic bi-preferences*. While the former model is identical to what we define here as preference structures, Giarlotta and Watson (2020) instead focus on exploring the structure of the latter model.

The main goal of the present paper is to explore how preference structures could be used to model the "choices" of an agent in a way that generalizes the classical rational choice theory. We are not aware of any work that studies this issue.

3.2 Examples

We next demonstrate the breadth of the model of preference structures.

Example 3.1. Let $\succsim$ be a complete preorder on a nonempty set X. Then, $(\succsim, \succsim)$ is a transitive preference structure on X. (Every complete preference relation may thus be thought of as a preference structure.)

Example 3.2. Let $\mathbf{R}$ be a total binary relation on a nonempty set X. Then, $(\triangle_X, \mathbf{R})$ is a preference structure on X. (Every total binary relation may thus be thought of as a preference structure.)

Example 3.3. Let $\succsim$ be a preorder on a nonempty set X. If $\mathbf{R}$ stands for $\succsim \cup \operatorname{Inc}(\succsim)$, then $(\succsim, \mathbf{R})$ is a preference structure on X. (This is, essentially, the model Mandler (2005) has considered in the context of consumer choice.)

Example 3.4. (*Aggregation by Social Welfare Criteria*) Fix a positive integer n, and let u_i be a real map on a nonempty set X for each $i = 1, ..., n$. Define a preorder $\succsim$ by $x \succsim y$ iff $u_i(x) \geq u_i(y)$ for each $i = 1, ..., n$, and the binary relations $\mathbf{R}^1$, $\mathbf{R}^2$, and $\mathbf{R}^3$ on X by

$$x \, \mathbf{R}^1 \, y \quad \text{iff} \quad \sum_{i=1}^{n} u_i(x) \geq \sum_{i=1}^{n} u_i(y),$$

$$x \, \mathbf{R}^2 \, y \quad \text{iff} \quad \min_{i=1,...,n} u_i(x) \geq \min_{i=1,...,n} u_i(y),$$

and

$$x \, \mathbf{R}^3 \, y \quad \text{iff} \quad \max_{i=1,...,n} (u_i(x) - u_i(y)) \geq 0,$$

respectively. We may think of $\succsim$ here as a *Pareto ordering*, while $\mathbf{R}^1$ and $\mathbf{R}^2$ correspond to the *utilitarian* and *Rawlsian* social welfare criteria, respectively. By contrast, $\mathbf{R}^3$, which often is nontransitive, is best viewed as a *justifiable preference* (to borrow the jargon used by Lehrer and Teper (2011)). It is readily checked that each $(\succsim, \mathbf{R}^i)$ is a weak preference structure on X. In fact, $(\succsim, \mathbf{R}^1)$ is a transitive preference structure, while $(\succsim, \mathbf{R}^2)$ is a transitive weak preference structure (but it need not be a preference structure). By contrast, $(\succsim, \mathbf{R}^3)$ need not be a preference structure, nor need it be transitive.

Example 3.5. (Aggregation by Majority Voting) Let $\mathcal{P}$ be a nonempty finite family of total preorders on a nonempty set X. Then, $\bigcap \mathcal{P}$ is the Pareto ordering induced by this collection. In turn, we define the (*majority voting*) binary relation $\mathcal{P}_{\text{maj}}$ on X as

$$x \, \mathcal{P}_{\text{maj}} \, y \quad \text{iff} \quad |\{\succsim \in \mathcal{P} : x \succ y\}| \geq |\{\succsim \in \mathcal{P} : y \succ x\}|$$

for every $x, y \in X$. Then, $(\bigcap \mathcal{P}, \mathcal{P}_{\text{maj}})$ is a preference structure on X.

Example 3.6. (Cautious Expected Utility Theory) Let I be a nonempty compact interval, and X the collection of all Borel probability measures on I. Take any nonempty collections $\mathcal{U}$ and $\mathcal{V}$ of continuous and strictly increasing real maps on I, and define the binary relation $\succsim_{\mathcal{U}}$ on X with

$$p \succsim_{\mathcal{U}} q \quad \text{iff} \quad \int_X u dp \geq \int_X u dq \text{ for every } u \in \mathcal{U}.$$

In the terminology of Dubra, Maccheroni and Ok (2004), $\succsim$ is a preorder on X that admits an expected multi-utility representation. Next, consider the binary relation $\mathbf{R}_{\mathcal{V}}$ on X defined by

$$p \, \mathbf{R}_{\mathcal{V}} \, q \quad \text{iff} \quad \inf_{u \in \mathcal{V}} \left(\int_X u dp \right) \geq \inf_{u \in \mathcal{V}} \left(\int_X u dq \right).$$

In the terminology of Cerreia-Vioglio, Dillenberger and Ortoleva (2015), $\mathbf{R}_{\mathcal{V}}$ is a complete preorder on X that admits a cautious expected utility representation. When these two representations use the same set of utilities, they become consistent. That is, $(\succsim_{\mathcal{U}}, \mathbf{R}_{\mathcal{U}})$ is a transitive weak preference structure on X. This need not be a preference structure, however.

Example 3.7. (Preferences with Imperfect Discrimination) Let $\mathbf{R}$ be a complete and quasitransitive binary relation on a nonempty set X. Then, $(\triangle_X \sqcup \mathbf{R}^{>}, \mathbf{R})$ is a preference structure on X. This model allows us to capture the utility model of imperfect discrimination which goes back to Armstrong (1939) and Luce (1956), and is studied more recently by Beja and Gilboa (1992), among others. To wit, let $u : X \to \mathbb{R}$ be any function and take any real number $\varepsilon \geq 0$. Define the binary relation $\mathbf{R}$ on X as $x\ \mathbf{R}\ y$ iff $u(x) \geq u(y) - \varepsilon$. This is a complete and quasitransitive binary relation on X with $x\ \mathbf{R}^{>}\ y$ iff $u(x) > u(y) + \varepsilon$ and $x\ \mathbf{R}^{=}\ y$ iff $|u(x) - u(y)| \leq \varepsilon$. Then, $(\succsim, \mathbf{R})$ is a preference structure on X where $\succsim$ is the semiorder on X defined by $x \succsim y$ iff either $x = y$ or $u(x) > u(y) + \varepsilon$. The interpretation is that the pairwise ranking of any two alternatives is an "easy" one if the utilities of these alternatives are sufficiently distinct, and "hard" otherwise.

Example 3.8. (*Intra-Dimensional Comparison Heuristics*) Let n be any positive integer, and consider an environment in which every commodity is modeled through n attributes. We thus put $X := \mathbb{R}^n$, and interpret any $x := (x_1, ..., x_n) \in X$ as a commodity which possesses x_i units of the attribute i. For each $i \in \{1, ..., n\}$, let us pick any skew-symmetric function $f_i : \mathbb{R}^2 \to (-1, 1)$ that is strictly increasing in the first component, and any strictly increasing and odd map $W : (-1, 1)^n \to \mathbb{R}$. We define the binary relation $\mathbf{R}$ on X as

$$x\,\mathbf{R}\,y \quad \text{iff} \quad W\left(f_1(x_1, y_1), ..., f_n(x_n, y_n)\right) \geq 0.$$

Here the vector $(f_1(x_1, y_1), ..., f_n(x_n, y_n))$ corresponds to comparisons of the goods x and y attribute by attribute; we can interpret f_i as measuring either the (dis)similarity of x_i and y_i or the salience of the ith attribute relative to the other attributes. We thus follow Tserenjigmid (2015), who has recently worked out a nice axiomatization for it, by calling $\mathbf{R}$ an *intra-dimensional comparison* (IDC) *relation*. Such relations contains the model of regret preferences of Loomes and Sugden (1982), the additive utility model (Example 3.5), the additive difference model of Tversky (1969) and a version of the salience theory of Bordalo, Gennaioli and Schleifer (2012). The upshot here is that $(\succsim, \mathbf{R})$ is a preference structure on X, where $\succsim$ is the binary relation on X defined by $x \succsim y$ iff $f_i(x_i, y_i) \geq 0$ for each $i = 1, ..., n$.

3.3 Weak vs. Proper Preference Structures

By definition, the strict part of the revealed preferences of a given preference structure extends the strict part of the core preference relation of that structure. A weak preference structure may fail this property, thereby not qualifying to be a preference structure. (See Examples 3.4 and 3.6.) But this is the only thing that separates a weak preference structure from a preference structure. That is, if $(\succsim, \mathbf{R})$ is a weak preference structure on X, the only reason why this may not be a preference structure is that there may be alternatives x and y in X such that the agent inherently prefers x over y strictly (that is, $x \succ y$) and yet the revealed preference $\mathbf{R}$ views x and y equally desirable (that is, $x \mathbf{R}^= y$). If, therefore, we refine $\mathbf{R}$ so as to drop (y, x) from it (for any such x and y), we would obtain a preference structure. Put differently, there is a natural way of assigning a preference structure to any weak preference structure.

To formalize this discussion, let $(\succsim, \mathbf{R})$ be a weak preference structure on X. We define the binary relation $\mathbf{R}_\succsim$ on X as follows: $x \, \mathbf{R}_\succsim \, y$ iff

$$\text{either } x \succsim y \text{ or } [x \text{ and } y \text{ are not } \succsim\text{-comparable and } x \, \mathbf{R} \, y]. \tag{1}$$

In words, the ranking of any two alternatives by $\mathbf{R}_\succsim$ is done lexicographically. We first check if the core relation $\succsim$ applies, invoking $\mathbf{R}$ only when $\succsim$ is unable to rank the alternatives (which we interpret as when the agent have difficulties in comparing x and y). While this is not obvious, $(\succsim, \mathbf{R}_\succsim)$ is indeed a preference structure on X.

Proposition 3.1. *Let $(\succsim, \mathbf{R})$ be a weak preference structure on a nonempty set X. Then, $(\succsim, \mathbf{R}_\succsim)$ is a preference structure on X.*

The Natural Epimorphism. We can also look at the situation from a categorical point of view. Let $\mathrm{w}\mathcal{PS}_X$ stand for the set of all weak preference structures on X, and $\mathcal{PS}_X$ for the set of all preference structures on X. Then, the map $\pi : \mathrm{w}\mathcal{PS}_X \to \mathcal{PS}_X$ defined by $\pi(\succsim, \mathbf{R}) := (\succsim, \mathbf{R}_\succsim)$, is a surjection that acts as the identity on $\mathcal{PS}_X$; we refer to π as the *natural epimorphism.* This map partitions $\mathrm{w}\mathcal{PS}_X$ in such a way that any one cell of the partition contains exactly one preference structure (which can be used as the representative of that cell). And it is "natural" in that, as we will see in Section 4, $(\succsim, \mathbf{R})$ and $(\succsim, \mathbf{R}_\succsim)$ are equivalent from the viewpoint of choice theory, at least as we develop it here.

A transitive weak preference structure may well fail to be a preference structure (Examples 3.4 and 3.6). Interestingly, the image of such a structure under the natural epimorphism, which, by Proposition 3.1, is a preference structure, may lose its transitivity. However, again in terms of the choice theory that we will introduce in Section 4, this does not make a difference, because the revealed preference part of such an image is sure to be quasitransitive.

Corollary 3.2. *Let $(\succsim, \mathbf{R})$ be a transitive weak preference structure on a nonempty set X. Then, $(\succsim, \mathbf{R}_\succsim)$ is a preference structure on X and $\mathbf{R}_\succsim$ is quasitransitive.*

Modeling Choice by Consultation. There is a nice interpretation of the image of a transitive weak preference structure under the natural epimorphism. Think of an individual who, when faced with a "hard" choice problem, seeks the advise of a consultant. The issue of dealing with "easy" choices is modeled by means of her (core) preference relation $\succsim$ on X. When two alternatives x and y are incomparable with respect to $\succsim$ – this choice is "hard" for the agent – she acts according to the advice of another individual (consultant). We imagine that the consultant is rational in the traditional sense, so her advice stems from a complete preorder $\mathbf{R}$ on X. Moreover, we assume that this preorder is consistent with $\succsim$ in the sense that $\succsim \subseteq \mathbf{R}$. (Otherwise, it would be unrealistic to presume that the agent trust the recommendations of the advisor, as some of those would conflict with her core preferences.) As such, $(\succsim, \mathbf{R})$ is a transitive weak preference structure on X. But, in this interpretation, $\mathbf{R}$ does not really correspond to the revealed preferences of the subject agent. After all, these preferences reflect those of the consultant only over the problems that this agent finds "hard," and thus seeks help for. In other words, the revealed preferences of the agent should coincide with $\succsim$ whenever $\succsim$ is able to render a ranking, and with $\mathbf{R}$ when this is not possible. Thus, the preference structure $\pi(\succsim, \mathbf{R})$ seems like the "right" model that corresponds to this interpretation. In this model, the revealed preferences of the agent need not be transitive, but they are quasitransitive.

3.4 Characterization of Preference Structures

The following result, in which X is an arbitrary nonempty set, provides a representation that connects the two components of a preference structure by means of a single entity, namely, a collection of preorders.

Theorem 3.3. *Let $\succsim$ and $\mathbf{R}$ be binary relations on X. Then, $(\succsim, \mathbf{R})$ is a (weak) preference structure on X if, and only if, there is a nonempty collection $\mathcal{P}$ of preorders on X such that*

$$(\succsim, \mathbf{R}) = \left(\bigcap \mathcal{P}, \bigcup \mathcal{P}\right) \tag{2}$$

where $\bigcup \mathcal{P}$ is complete and each $\trianglerighteq \in \mathcal{P}$ extends (includes) $\succsim$.[6]

The "if' part of this result provides a general method of defining preference structures. In turn, its "only if" part provides a *multi-selves* interpretation for any given preference structure

[6] A similar result for necessary and possible preferences was given by Giarlotta and Greco (2013).

$(\succsim, \mathbf{R})$. To wit, let $\mathcal{P}$ stand for a nonempty collection of preorders on X as found in Theorem 3.3. We may think of each element $\trianglerighteq$ in $\mathcal{P}$ as a (potentially incomplete) preference relation of a different "self" of the same individual. (For instance, the agent may not know which of these relations will be the relevant one at the time of consumption, so entertains them all before making her choice.) These "selves" of the agent are consistent with her core preference relation $\succsim$ as every one of them extends $\succsim$. In addition, $\succsim$, being equal to $\bigcap \mathcal{P}$, ranks an alternative x over another alternative y iff every one of her "selves" agrees that this is the correct ranking; $\succsim$ may thus be thought of as a *unanimity* relation. On the other hand, the revealed preference relation $\mathbf{R}$ of the agent, being equal to $\bigcup \mathcal{P}$, ranks x over y iff at least one of her "selves" agrees that this is the correct ranking. In this sense, we may think of $\mathbf{R}$ as a *rationalizable* preference on X, borrowing (and slightly abusing) the terminology used by Cherepanov, Feddersen and Sandroni (2013). Importantly, these notions of unanimity and rationalizability are compatible, for they are based on the preferences of the same set of "selves" of the agent.[7]

We conclude by noting that Theorem 3.3 modifies readily to give a characterization of transitive preference structures. This reads particularly simple if we impose the completeness assumption (for the second relation) at the outset:

Corollary 3.4. *Let* $\succsim$ *and* $\mathbf{R}$ *be binary relations on* X *with* $\mathbf{R}$ *being complete. Then,* $(\succsim, \mathbf{R})$ *is a transitive* (*weak*) *preference structure on* X *if, and only if, there is a nonempty collection* $\mathcal{P}$ *of preorders on* X *such that* (2) *holds,* $\mathbf{R} \in \mathcal{P}$*, and each* $\trianglerighteq \in \mathcal{P}$ *extends* (*includes*) $\succsim$.

4 Choice by Preference Structures

In this section, we begin to analyze how "choices" may arise from preference structures. This necessitates that we agree on what it means for an alternative to "maximize" a given complete (but not necessarily transitive) binary relation on a given feasible set, so we start the section with the discussion of this issue.

[7]The reader may ask if one can guarantee the completeness of each member of $\mathcal{P}$ in the representation provided in Theorem 3.3. Unfortunately, this is a very restrictive requirement; we can do this only when $\mathbf{R}$ is obtained from $\succsim$ by rendering every $\succsim$-incomparable pair indifferent. That is, a preference structure $(\succsim, \mathbf{R})$ on X satisfies $\mathbf{R} = \succsim \sqcup \mathrm{Inc}(\succsim)$ iff there is a nonempty collection $\mathcal{P}$ of complete preorders on X such that (i) $(\succsim, \mathbf{R}) = (\bigcap \mathcal{P}, \bigcup \mathcal{P})$ and (ii) each $\trianglerighteq \in \mathcal{P}$ extends $\succsim$. (We omit the proof, which is available upon request.)

4.1 Maximization of Complete Binary Relations

Let X be a nonempty set, $\mathbf{R}$ a binary relation on X, and S a nonempty subset of X. An element x of S is called $\mathbf{R}$-*maximal* in S if there is no $y \in S$ with $y \, \mathbf{R}^{>} \, x$, and $\mathbf{R}$-*maximum* in S if $x \, \mathbf{R} \, S$. We denote the set of all $\mathbf{R}$-maximal and $\mathbf{R}$-maximum elements in S by $\mathbf{MAX}(S, \mathbf{R})$ and $\max(S, \mathbf{R})$, respectively. We always have $\max(S, \mathbf{R}) \subseteq \mathbf{MAX}(S, \mathbf{R})$, but this inequality may hold strictly (unless $\mathbf{R}$ is complete).

For a complete, but nontransitive, binary relation $\mathbf{R}$, $\mathbf{MAX}(S, \mathbf{R})$ may be empty even for a finite set S. For this reason, alternative notions of extrema are developed for binary relations. The best-known of these is the notion of *top-cycles*.

Top-Cycles. Let $\mathbf{R}$ be a complete binary relation on X. We say that a nonempty subset A of S is a *highset in S with respect to* $\mathbf{R}$, or more simply, an $\mathbf{R}$-*highset in S*, if

$$x \, \mathbf{R}^{>} \, y \quad \text{for every } x \in A \text{ and } y \in S \setminus A.$$

Notice that the collection of all $\mathbf{R}$-highsets in S is nonempty, because it contains S. Moreover, this collection is linearly ordered by set inclusion $\supseteq$.[8] Consequently, if it exists, there is a unique smallest $\mathbf{R}$-highset in S, namely, the intersection of all $\mathbf{R}$-highsets in S. We thus define the *top-cycle in S with respect to* $\mathbf{R}$ as

$$\bigcirc(S, \mathbf{R}) := \bigcap \{A : A \text{ is an } \mathbf{R}\text{-highset in } S\}.$$

This set is nonempty iff the smallest $\mathbf{R}$-highset in S exists. In particular, $\bigcirc(S, \mathbf{R}) \neq \emptyset$ whenever S is a nonempty finite set.[9]

By "maximization of $\mathbf{R}$ in S," we mean identifying $\bigcirc(S, \mathbf{R})$. This is not only intuitive, but it is also consistent with the standard case (because $\bigcirc(S, \mathbf{R})$ reduces to $\max(S, \mathbf{R})$ when $\mathbf{R}$ is transitive). Furthermore, the following fact demonstrates that top-cycles indeed correspond to a well-defined optimization principle, thereby also clarifying that our definition is consistent with how top-cycles are traditionally defined in, say, social choice theory.

---

[8] *Proof.* Suppose A and B are two $\mathbf{R}$-highsets in S with $A \subseteq B$ false. Then, pick any $a \in A \setminus B$, and notice that, for any $b \in B$, we have $b \, \mathbf{R}^{>} \, a$ because $a \in S \setminus B$ and B is an $\mathbf{R}$-highset in S. As A is itself an $\mathbf{R}$-highset in S, and $a \in A$, this implies $b \in A$ for each $b \in B$.

[9] Top-cycles are studied extensively in the case where $\mathbf{R}$ is a tournament (that is, an asymmetric total binary relation on a finite set). See, for instance, Laslier (1997).

Proposition 4.1. *Let S be a nonempty subset of a set X, and $\mathbf{R}$ a complete binary relation on X. Then,*

$$\bigcirc(S, \mathbf{R}) = \max(S, \mathrm{tran}(\mathbf{R}|_S)).$$

Let us call a nonempty subset A of X an **R-cycle** if for any $x, y \in A$, there are finitely many $a_1, ..., a_k \in A$ with $x \mathbf{R} a_1 \mathbf{R} \cdots \mathbf{R} a_k \mathbf{R} y$. (If A is finite and $\mathbf{R}$ is a complete binary relation on X, then A is an $\mathbf{R}$-cycle iff we can enumerate A as $\{x_1, ..., x_n\}$ so that $x_1 \mathbf{R} x_2 \mathbf{R} \cdots \mathbf{R} x_n \mathbf{R} x_1$.) We can use Proposition 4.1 to obtain another useful characterization of top-cycles. The next result, which generalizes a theorem of Schwartz (1972), identifies $\bigcirc(S, \mathbf{R})$ as the unique $\mathbf{R}$-highset in S that is also an $\mathbf{R}$-cycle.

Corollary 4.2. *Let S and T be nonempty subsets of X, and $\mathbf{R}$ a complete binary relation on X such that $\bigcirc(S, \mathbf{R}) \neq \emptyset$. Then, $T = \bigcirc(S, \mathbf{R})$ if, and only if, T is both an $\mathbf{R}$-highset in S and an $\mathbf{R}$-cycle.*

Proof. Put $T := \bigcirc(S, \mathbf{R})$. Then, T is an $\mathbf{R}$-highset in S. Further, by Proposition 4.1, if $x, y \in T$, then $x \, \mathrm{tran}(\mathbf{R}|_S)^= y$, so there exist $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in S$ with $x \mathbf{R} a_1 \mathbf{R} \cdots \mathbf{R} a_k \mathbf{R} y$, that is, T is an $\mathbf{R}$-cycle. Conversely, suppose that T is both an $\mathbf{R}$-highset in S and an $\mathbf{R}$-cycle. The first of these hypotheses implies $x \, \mathrm{tran}(\mathbf{R}|_S)^> y$ for any $x \in T$ and $y \in S \backslash T$, whereas the second implies $x \, \mathrm{tran}(\mathbf{R}|_S)^= y$ for all $x, y \in T$. Put together, we get $T = \max(S, \mathrm{tran}(\mathbf{R}|_S))$. In view of Proposition 4.1, we are done. ■

4.2 Rationalization by Preference Structures

We now turn to the primary inquiry of the present paper, namely, to the issue of defining how "choices" are made on the basis of a given preference structure. To this end, let X be any nonempty set, and let $\mathfrak{X}$ be any collection of nonempty subsets of X such that (i) $\mathfrak{X}$ contains all singletons, and (ii) $\mathfrak{X}$ is closed under taking finite unions. (In particular, $\mathfrak{X}$ contains all nonempty finite subsets of X). For ease of reference, we will refer to any such ordered pair $(X, \mathfrak{X})$ as a *choice environment*. For example, $(X, 2^X \backslash \{\emptyset\})$ is a choice environment. More generally, where $\mathfrak{X}_{<\infty}$ denotes the collection of all nonempty finite subsets of X, $(X, \mathfrak{X}_{<\infty})$ is a choice environment; this is the environment used by the vast majority of works in the theory of individual choice. Still more generally, $(X, \mathbf{k}(X))$ is a choice environment, where X is a topological space and $\mathbf{k}(X)$ stands for the set of all nonempty compact subsets of X.

Given any choice environment $(X, \mathfrak{X})$, by a *choice correspondence on* $\mathfrak{X}$, we mean a set-valued map $C : \mathfrak{X} \rightrightarrows X$ such that $C(S) \subseteq S$ for every $S \in \mathfrak{X}$ and $C(S) \neq \emptyset$ for every finite $S \in \mathfrak{X}$. Such a choice correspondence C is said to be *single-valued* if $|C(S)| = 1$ for every finite $S \in \mathfrak{X}$.

Now take any weak preference structure $(\succsim, \mathbf{R})$ on X. We say that a choice correspondence C on $\mathfrak{X}$ is *rationalized by* $(\succsim, \mathbf{R})$ if

$$C(S) = \bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}), \tag{3}$$

or equivalently,

$$C(S) = \max(\mathbf{MAX}(S, \succsim), \text{tran}(\mathbf{R}|_{\text{MAX}(S,\succsim)})), \tag{4}$$

for every $S \in \mathfrak{X}$. This maintains that a rational agent with a weak preference structure $(\succsim, \mathbf{R})$ makes her choice(s) from a given feasible set S by employing a two-step procedure. First, she looks for those alternatives in S that are maximal with respect to her core preference relation $\succsim$. If there is only one such alternative in S, then she chooses that alternative. If there is a multiplicity of such alternatives (which may be due to indifferences and/or incomparabilities instigated by $\succsim$), she restricts her attention to those alternatives, and evaluates them on the basis of her second (complete) binary relation $\mathbf{R}$. She finalizes her choice(s) by maximizing $\mathbf{R}$ on $\mathbf{MAX}(S, \succsim)$ in the sense of finding the top-cycle in $\mathbf{MAX}(S, \succsim)$ with respect to $\mathbf{R}$. This top-cycle is the set of all alternatives she deems "choosable" in S.

Remark. After the important contribution of Manzini and Mariotti (2007), choice correspondences $S \mapsto \max(\mathbf{MAX}(S, \mathbf{R}_1), \mathbf{R}_2)$, where $\mathbf{R}_1$ and $\mathbf{R}_2$ are binary relations on a finite set X, are often called *sequentially rationalized choice procedures*. The spirit of choice correspondences rationalized by preference structures is certainly in concert with such procedures. And yet, formally speaking, the latter type of a choice correspondence need not be a sequentially rationalized choice procedure. Indeed, for a feasible set $S \in \mathfrak{X}$, $\text{tran}(\mathbf{R}|_{\text{MAX}(S,\succsim)})$ is not the same relation as $\text{tran}(\mathbf{R})$ in general. Thus, there is no "one" second binary relation used in a choice correspondence rationalized by a preference structure.[10]

[10]When X is finite and $S, T \in \mathfrak{X}$, we have $C(S) \cap C(T) \subseteq C(S \cup T)$ for any sequentially rationalized choice correspondence C on $\mathfrak{X}$ (García-Sanz and Alcantud (2015)). But if C is rationalized by a preference structure, it need not obey this property. For instance, let $X := \{x_1, ..., x_5\}$, $\succsim := \triangle_X \sqcup \{(x_3, x_5), (x_4, x_2)\}$, and consider the total binary relation $\mathbf{R}$ on X with $\mathbf{R}^{>} := \{(x_3, x_5), (x_4, x_2), (x_3, x_1), (x_4, x_1)\}$. Then, $(\succsim, \mathbf{R})$ is a preference structure on X. Now, define C by (3), put $S := \{x_1, x_2, x_3\}$ and $T := \{x_1, x_4, x_5\}$, and check that x_1 belongs to $C(S) \cap C(T)$, but not to $C(S \cup T)$.

4.3 Existence of Choice by Preference Structures

One of the main principles of optimization theory is the fact there is a maximum element in every compact space with respect to a continuous and complete preorder. In this section, we show that this principle extends to the more general context of preference structures.

We say that a binary relation on a topological space X is *continuous* if it is a closed subset of $X \times X$ (relative to the product topology). In turn, a weak preference structure $(\succsim, \mathbf{R})$ is *continuous* if both $\succsim$ and $\mathbf{R}$ are continuous binary relations on X. Our main existence theorem says that $\bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}) \neq \emptyset$ for any such $(\succsim, \mathbf{R})$ and nonempty compact $S \subseteq X$.

Theorem 4.3. *For any topological space X, the choice correspondence on $\mathbf{k}(X)$ rationalized by a continuous weak preference structure $(\succsim, \mathbf{R})$ on X is nonempty-valued.*

One of the major difficulties with working with nontransitive preferences in an economic setting is that even finite menus may not possess a maximal element with respect to such a preference. Theorem 4.3 shows that the present theory is free from this difficulty. Under the usual assumptions of compactness and continuity, there always exists a "choice" with respect to a preference structure, provided that we define choice by a top-cycle element (with respect to one's revealed preference) within the set of all maximal elements in a menu (with respect to that person's core preference).

The earliest topological existence theorem for top-cycles that we know is due to Kalai and Schmeidler (1977). While that theorem applies to complete and continuous binary relations only under the hypothesis of antisymmetry, Duggan (2007) showed that the antisymmetry requirement is in fact not needed. In turn, Theorem 4.3 generalizes Duggan's Theorem; the latter obtains from Theorem 4.3 simply by setting $\succsim = \triangle_X$.

In passing, we note that Theorem 4.3 cannot be proved by applying Duggan's Theorem to $\mathbf{MAX}(S, \succsim)$, where S is a compact subset of topological space X and $\succsim$ a continuous preorder on X. Indeed, these hypotheses do not guarantee that $\mathbf{MAX}(S, \succsim)$ is compact. We prove Theorem 4.3 in the Appendix by means of a direct argument that does not involve Duggan's Theorem.

4.4 Equivalent Preference Structures

In the standard theory of rational choice, a choice correspondence can be rationalized by at most one complete preference relation (provided that the domain of the correspondence is rich enough). This is not true for choice correspondences that are rationalized by preference structures. That is, more than one preference structure may well rationalize a given choice correspondence. We may think of such structures as "equivalent" from the perspective of choice.

Definition. Given any choice environment $(X, \mathfrak{X})$, two weak preference structures $(\succsim, \mathbf{R})$ and $(\succsim', \mathbf{R}')$ on X are said to be *equivalent* if

$$\bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}) = \bigcirc(\mathbf{MAX}(S, \succsim'), \mathbf{R}')$$

for every $S \in \mathfrak{X}$; we denote this situation by writing $(\succsim, \mathbf{R}) \cong (\succsim', \mathbf{R}')$.

In Section 3.3, we have stated that there is a "natural" way of pruning a weak preference structure to make it a proper preference structure that is indistinguishable from the former in terms of choice theory. The above notion of *equivalence* helps formalize this point. Indeed, where $(\succsim, \mathbf{R})$ is a weak preference structure on X, and where $\mathbf{R}_{\succsim}$ is defined by (1), we are sure to have $(\succsim, \mathbf{R}) \cong (\succsim, \mathbf{R}_{\succsim})$.

Proposition 4.5. *In the context of any choice environment* $(X, \mathfrak{X})$, *any weak preference structure* $(\succsim, \mathbf{R})$ *on* X *is equivalent to the (proper) preference structure* $(\succsim, \mathbf{R}_{\succsim})$.

Proof. Take any $S \in \mathfrak{X}$, and note that there is nothing to prove if there is no $\succsim$-maximal element in S. So, assume otherwise, and take any $\succsim$-maximal elements x and y of S. Then, either $x \sim y$ or $(x, y) \in \text{Inc}(\succsim)$. In the former case, we have $x\,\mathbf{R}^{=}\,y$ and $x\,\mathbf{R}_{\succsim}^{=}\,y$ (because both $\mathbf{R}$ and $\mathbf{R}_{\succsim}$ are superrelations of $\succsim$), and in the latter case, $x\,\mathbf{R}\,y$ iff $x\,\mathbf{R}_{\succsim}\,y$ by definition of $\mathbf{R}_{\succsim}$. Thus, the restrictions of $\mathbf{R}$ and $\mathbf{R}_{\succsim}$ to $\mathbf{MAX}(S, \succsim)$ are the same. In view of the arbitrary choice of S, and Proposition 4.1, therefore, $(\succsim, \mathbf{R}) \cong (\succsim, \mathbf{R}_{\succsim})$. ■

Let $(X, \mathfrak{X})$ be a choice environment, and recall that $\text{w}\mathcal{PS}_X$ and $\mathcal{PS}_X$ stand for the set of all weak and proper preference structures on X, respectively, while the natural epimorphism $\pi : \text{w}\mathcal{PS}_X \to \mathcal{PS}_X$ is defined by $\pi(\succsim, \mathbf{R}) := (\succsim, \mathbf{R}_{\succsim})$. Now, let $\text{Choice}_{\mathfrak{X}}$ denote the family of all choice correspondences on $\mathfrak{X}$, and define the map $c : \text{w}\mathcal{PS}_X \to \text{Choice}_{\mathfrak{X}}$ by setting $c(\succsim, \mathbf{R})$ to be the choice correspondence on $\mathfrak{X}$ that is rationalized by $(\succsim, \mathbf{R})$. Then, Proposition 4.5 says that the following diagram, in which the restriction of c to $\mathcal{PS}_X$ is also denoted by c, commutes. In particular, π is a selection from the quotient map on $\text{w}\mathcal{PS}_X$ relative to the equivalence relation $\cong$.

The main takeaway here is that a choice correspondence is rationalizable by a weak preference structure if, and only if, it is rationalizable by a proper preference structure.

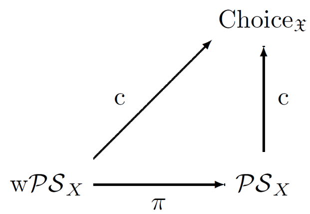

Figure 4.1

4.5 Examples

The following examples are meant to illustrate the breadth of the notion of rationalization by preference structures. Unless otherwise is explicitly stated, $(X, \mathfrak{X})$ stands below for an arbitrarily fixed choice environment.

Example 4.1. (*The Rational Choice Model*) Let $\succsim$ be a complete preorder on X. Then, the choice correspondence C on $\mathfrak{X}$ rationalized by the transitive preference structure $(\succsim, \succsim)$ satisfies

$$C(S) = \max(S, \succsim) \quad \text{for every } S \in \mathfrak{X}.$$

Thus, the choice theory based on preference structures generalizes the standard choice theory that is based on complete preference relations.

Example 4.2. (*The Top-Cycle Choice Rule*) For a complete binary relation $\mathbf{R}$ on X, the choice correspondence C on $\mathfrak{X}$ rationalized by the preference structure $(\triangle_X, \mathbf{R})$ satisfies

$$C(S) = \bigcirc(S, \mathbf{R}) \quad \text{for every } S \in \mathfrak{X}.$$

Thus, the choice theory based on preference structures generalizes the theory of top-cycle choice rules that are commonly used in the theory of social choice and tournaments (cf. Kalai and Schmeidler (1977), Schwartz (1986), Laslier (1997), and Ehlers and Sprumont (2008).)

Example 4.3. (*The Undominated Choice Rule*) Let $\succsim$ be a preorder on X. Then, the choice correspondence C on $\mathfrak{X}$ rationalized by the preference structure $(\succsim, \succsim \sqcup \operatorname{Inc}(\succsim))$ satisfies

$$C(S) = \mathbf{MAX}(S, \succsim) \quad \text{for every } S \in \mathfrak{X}.$$

Thus, the choice theory based on preference structures generalizes the choice theory that is based on incomplete (but transitive) preference relations. (See, for instance, Eliaz and Ok (2006).)

Example 4.4. (*Pareto Refinement of Majority Voting*) Let $\mathcal{P}$ and $\mathcal{P}_{\text{maj}}$ be defined as in Example 3.5. Then, the choice correspondence C on $\mathfrak{X}$ rationalized by the preference structure $(\bigcap \mathcal{P}, \mathcal{P}_{\text{maj}})$ assigns to any feasible set $S \in \mathfrak{X}$ those Pareto optimal outcomes in S that maximize the transitive closure of the majority voting rule on S. (Here, of course, Pareto optimality and majority voting rule are understood relative to the preference relations in $\mathcal{P}$.)

Example 4.5. (*Transitive Preference Structures*) We have noted in Section 4.2 that a choice correspondence rationalized by a preference structure is, in general, not a sequential choice procedure (in the sense of Manzini and Mariotti (2007)). However, the situation is different in the transitive case. To wit, let $(\succsim, \mathbf{R})$ be a transitive weak preference structure on a nonempty set X, and let C stand for the choice correspondence on $\mathfrak{X}$. Then,

$$C(S) = \max(\mathbf{MAX}(S, \succsim), \mathbf{R}) \tag{5}$$

for every $S \in \mathfrak{X}$. This sits square with the interpretation that $\succsim$ is the "sure" preferences of a person, and $\mathbf{R}$ corresponds to the rational (complete) preferences of a consultant. When dealing with a choice problem S, this person first identifies the undominated alternatives in S with respect to her inherent (core) preference relation $\succsim$. If there are more than one such alternative in S, then she is conflicted as to which of these to choose. In that case, she presents her reduced choice problem $\mathbf{MAX}(S, \succsim)$ to her consultant who identifies the best alternatives within $\mathbf{MAX}(S, \succsim)$ according to her own preferences, and our principal agent chooses (one of those) alternatives.

At the end of Section 3.3, we have noted that "choice-by-consultation" would be better modeled by means of the image of $(\succsim, \mathbf{R})$ under the natural epimorphism, that is, by $(\succsim, \mathbf{R}_{\succsim})$. The latter model is a proper preference structure, but it need not be transitive (Section 3.3). Nonetheless, we still have

$$C(S) = \max(\mathbf{MAX}(S, \succsim), \mathbf{R}_{\succsim}) \tag{6}$$

for every $S \in \mathfrak{X}$ with $C(S) \neq \emptyset$. (Indeed, for any such S, we have seen in the proof of Proposition 4.5 that $\mathbf{R}_{\succsim}$ agrees with $\mathbf{R}$, and hence it is transitive, on $\mathbf{MAX}(S, \succsim)$, so (6) follows from Proposition 4.1.)

Remark. In the context of Example 4.5, we have $C = \max(\cdot, \mathbf{R})$, provided that $(\succsim, \mathbf{R})$ is a transitive *proper* preference structure. This is not true if $(\succsim, \mathbf{R})$ is a transitive *weak* preference structure, however. In that case, we have $C \subseteq \max(\cdot, \mathbf{R})$ if X is finite (or more generally, if for every $S \in \mathfrak{X}$ and $y \in$

$S \setminus \mathbf{MAX}(S, \succsim)$, there is an $x \in \mathbf{MAX}(S, \succsim)$ with $x \succ y$). But easy examples would show that the converse containment need not hold even when X is finite.

Example 4.6. (*The Constant Threshold Choice Model*) Let $u : X \to \mathbb{R}$ be any function and take any real number $\varepsilon \geq 0$. Define the binary relation $\mathbf{R}$ on X as $x \mathbf{R} y$ iff $u(x) \geq u(y) - \varepsilon$. Consider first the preorder $\succsim'$ on X defined by $x \succsim' y$ iff either $x = y$ or $u(x) > u(y)$. Then, $(\succsim', \mathbf{R})$ is a weak preference structure on X, and the choice correspondence C on $\mathfrak{X}$ rationalized by this preference structure is the rational choice model: $C(S) = \arg\max\{u(x) : x \in S\}$ for every $S \in \mathfrak{X}$. Next, consider the preorder $\succsim$ on X defined by $x \succsim y$ iff either $x = y$ or $u(x) > u(y) + \varepsilon$. Then, $(\succsim, \mathbf{R})$ is a preference structure on X (Example 3.7), and it is easy to prove that the choice correspondence C on $\mathfrak{X}$ rationalized by $(\succsim, \mathbf{R})$ satisfies

$$C(S) = \{x \in S : \sup u(S) - u(x) \leq \varepsilon\}$$

for every $S \in \mathfrak{X}$. Following Luce (1956), such a correspondence is referred to as a *constant threshold choice model*. Conclusion: Every constant threshold choice model is rationalized by a preference structure.

4.6 On the Predictive Power of the Model

The examples above illustrate that many well-known choice models are indeed rationalized by preference structures. In this section, we go the other direction, and look at those choice correspondences that cannot be rationalized as such. In other words, our goal is to investigate the extent of the predictive power of our model. We begin with perhaps the simplest illustration of the fact that not every choice behavior abides by this notion of rationality.

Example 4.7. Put $X := \{x, y, z\}$, and take any choice correspondence C on $\mathfrak{X}_{<\infty}$ such that

$$x \in C\{x, y\} \quad \text{and} \quad \{y\} = C\{x, y, z\}.$$

To derive a contradiction, suppose C is a choice correspondence rationalized by a preference structure $(\succsim, \mathbf{R})$ on X. Since $y \in C\{x, y, z\}$, y is $\succsim$-maximal in X, and hence in $\{x, y\}$, so $x \in C\{x, y\}$ implies that $x \mathbf{R} y$ (Proposition 4.1). Since x does not belong to $C\{x, y, z\}$ but y does, therefore, x is not $\succsim$-maximal in X. As $y \succ x$ cannot hold (because $x \in C\{x, y\}$), we thus have $z \succ x$. Then, $y \succ z$ cannot hold (because $\succ$ is transitive). It follows that z is $\succsim$-maximal in X, and hence $y \mathbf{R}^{\succ} z$ (because $\{y\} = C\{x, y, z\}$). Thus, $z \succ x \mathbf{R} y$ and $y \mathbf{R}^{\succ} z$, which contradicts $\succsim$-transitivity of $\mathbf{R}$.

Weak Axiom of Revealed Preference. Let X be any nonempty set, and recall that $\mathfrak{X}_{<\infty}$ stands for the family of all nonempty finite subsets of X. The most well-known rationality criterion for a choice correspondence C on $\mathfrak{X}_{<\infty}$ is the *Weak Axiom of Revealed Preference* (*WARP*). To be precise, we say that C satisfies the *Property* α if $C(T) \cap S \subseteq C(S)$ for every $S, T \in \mathfrak{X}_{<\infty}$ with $S \subseteq T$, and that it satisfies *Property* β if for every $S, T \in \mathfrak{X}_{<\infty}$ with $S \subseteq T$, and $x, y \in C(S)$ with $x \in C(T)$, we have $y \in C(T)$. Finally, C is said to satisfy WARP if it obeys both of these properties. A fundamental result of choice theory says that C satisfies WARP iff it is rationalizable by a complete preference relation in the sense that there exists a total preorder $\succsim$ on X with $C(S) = \max(S, \succsim)$ for every $S \in \mathfrak{X}_{<\infty}$.

Unsurprisingly, a choice correspondence rationalized by a preference structure need not satisfy either Property α or Property β. After all, choice correspondences of the form considered in Examples 4.2 and 4.3 are well-known to fail Properties α and β, respectively. In fact, Example 4.3 shows that even a choice correspondence on $\mathfrak{X}_{<\infty}$ that is rationalized by transitive weak preference structure may violate Property β.

Single-Valued Choice Correspondences. While rationalization by a preference structure ensures, in general, neither Property α nor Property β, it turns out that it reduces to the standard notion of rationalization in the context of *single-valued* choice correspondences.

Proposition 4.6. *Let X be any nonempty set, and C a single-valued choice correspondence on $\mathfrak{X}_{<\infty}$. Then, C is rationalized by a weak preference structure if, and only if, it satisfies WARP.*

Proof. The "if" part of the claim is straightforward. Conversely, suppose C is rationalized by a weak preference structure $(\succsim, \mathbf{R})$ on X. Take any $S, T \in \mathfrak{X}_{<\infty}$ with $S \subseteq T$, and let $x \in S$ be such that $\{x\} = C(T)$. Then, $x \in \mathbf{MAX}(T, \succsim)$ and $x \mathbf{R} \mathbf{MAX}(T, \succsim)$. To derive a contradiction, suppose x does not belong to $C(S)$. Since $S \subseteq T$, we have $x \in \mathbf{MAX}(S, \succsim)$ so there must then exist a $y \in \mathbf{MAX}(S, \succsim)$ with $y \mathbf{R}^{>} x$. Since y does not belong to $C(T)$, therefore, y cannot be $\succsim$-maximal in T. Since T is finite, this means that $z \succ y$ for some $z \in \mathbf{MAX}(T, \succsim)$. Then, by $\succsim$-transitivity of $\mathbf{R}$, we find $z \mathbf{R} x$. By the choice of x, this is possible only if $z = x$. But this implies $x \succ y$, contradicting $\succsim$-maximality of y in S. Conclusion: C satisfies Property α. As for single-valued choice correspondence WARP is equivalent to Property α, we are done. ■

Proposition 4.6 is another demonstration of the predictive power of rationalization by preference structures. Indeed, it shows that the choice theory that is based on preference structures reduces to the standard theory of rational choice in the context of single-valued choice correspondences on finite choice problems. (Thus, for instance, the rational shortlisting models of Manzini and Mariotti (2007), Au and Kawai (2011), and Cherepanov et al. (2013), as well as the attention/competition filter model of Masatlioglu, Nakajima and Ozbay (2012), are not captured by this theory.) This is not really surprising. After all, the main goal of the model of preference structures is to capture behavioral traits such as indecisiveness and cyclic choices, and as such, the choice theory that this model induces is primed to make many-valued choice predictions.

The Condorcet Criterion. Let $(X, \mathfrak{X})$ be a choice environment. A choice correspondence C on $\mathfrak{X}$ is said to satisfy the **Condorcet Criterion** if for every $S \in \mathfrak{X}$ and $x \in S$,

$$x \in C\{x, y\} \text{ for every } y \in S \quad \text{imply} \quad x \in C(S).$$

The choice behavior that is rationalizable by a preference structure is sure to be consistent with this property.

Proposition 4.7. *Let C be a choice correspondence on $\mathfrak{X}$ rationalized by a weak preference structure $(\succsim, \mathbf{R})$ on X. Then, C satisfies the Condorcet Criterion.*

Proof. Take any $S \in \mathfrak{X}$, and let x be an element of S such that $x \in C\{x, y\}$ for every $y \in S$. Then, $x \in \mathbf{MAX}(S, \succsim)$ so, given that $C(S)$ is the top-cycle in $\mathbf{MAX}(S, \succsim)$ with respect to $\mathbf{R}$, if x did not belong to $C(S)$, we would have $y \mathbf{R}^{>} x$ for some $y \in \mathbf{MAX}(S, \succsim)$. But then, for this y, we would have $\{y\} = C\{x, y\}$, a contradiction. Conclusion: C satisfies the Condorcet Criterion. ■

Proposition 4.7 can also be used to distinguish those choice correspondences rationalized by preference structures from some of the boundedly rational choice correspondences introduced in the recent literature. For instance, we see readily that the reference-dependent choice model of Ok, Ortoleva and Riella (2015), or the model of choice with limited consideration, introduced by Lleras et al. (2017), are distinct from the present choice model as they do not satisfy the Condorcet Criterion. Similarly, justifiable choice correspondences are, in general, not rationalizable by a weak preference structure.[11]

[11] A choice correspondence C on $\mathfrak{X}$ is said to be *justified* by a collection $\mathcal{P}$ of complete preorders on X if $C(S) = \bigcup\{\max(S, \succsim) : \succsim \in \mathcal{P}\}$ for every $S \in \mathfrak{X}$. (See, for instance, Heller (2012) and Costa, Ramos and Riella (2020).) Now, let $X := \{x, y, z\}$, and consider the choice correspondence C on $\mathfrak{X}$ with $C(X) = \{x, z\}$ and $C(S) = S$ for each nonempty proper subset S of X. Then, C fails the Condorcet Criterion, but it is justified by $\{\succsim_1, \succsim_2\}$ where $x \succ_1 y \succ_1 z$ and $z \succ_2 y \succ_2 x$.

Finally, note that Example 4.6 shows that certain types of *satisficing* rules à la Herbert Simon are captured by the present choice theory. But not all threshold choice models, let alone all satisficing rules, are rationalizable by preference structures. For example, consider the choice correspondence C on $\mathfrak{X}$ defined by

$$C(S) = \{x \in S : \sup u(S) - u(x) \leq \varepsilon(S)\},$$

where u is any real map on X and $\varepsilon : \mathfrak{X} \to \mathbb{R}_+$ is a function such that $\varepsilon(A) \leq \varepsilon(B)$ for every $A, B \in \mathfrak{X}$ with $A \subseteq B$. (Where X is finite, Frick (2016) has provided an axiomatization of such choice correspondences, and called them *monotone threshold choice models*.) Such a model need not be rationalizable by a preference structure.

Characterization of Rationalization by Preference Structures. It is of great interest to provide a behavioral axiomatic characterization of all choice correspondences that are rationalizable by a preference structure. This would formally describe the predictive content of such correspondences fully, and provide a complete comparison of them with rational choice correspondences. Given the sizable length of the present paper, however, we do not take up this exercise here. Instead, this problem is tackled in a companion paper by Evren, Nishimura and Ok (2021), where a complete characterization is provided in the context of the choice environment $(X, \mathfrak{X}_{<\infty})$.

Other Behavioral Consistency Properties. Suppose C is a rationalizable choice correspondence on $\mathfrak{X}$ in the classical sense, that is, $C = \max(\cdot, \succsim)$ for some complete preorder $\succsim$ on X. Let z be a choice from a feasible set S by a decision maker whose choice behavior is modeled by this choice correspondence. If this agent is instead offered the feasible set $S \cup \{x\}$ where x is a new alternative at least as good as z, she would surely deem x choosable from this set: $x \in C(S \cup \{x\})$. It is in this sense that C is *monotonic* with respect to $\succsim$.

Now let C be the choice correspondence rationalized by some preference structure $(\succsim, \mathbf{R})$ on X. We would like to carry out the same query in this case as well, but now monotonicity may be checked either with respect to the core preference relation $\succsim$ of the agent, or her revealed preferences $\mathbf{R}$. Let us first look into the first situation. Suppose $z \in C(S)$ for some $S \in \mathfrak{X}$. Then, if the agent has no doubt in her mind that x is a better alternative than z, that is, $x \succsim z$, one would expect she view x as choosable from $S \cup \{x\}$. The following proposition shows that C possesses this property indeed.

Proposition 4.8. *Let C be a choice correspondence on $\mathfrak{X}$ rationalized by a weak preference structure $(\succsim, \mathbf{R})$ on X. Then, for any $S \in \mathfrak{X}$,*

$$x \succsim z \in C(S) \quad \textit{implies} \quad x \in C(S \cup \{x\}).$$

Let us now ask the same question with respect to the revealed preference relation $\mathbf{R}$. That is, suppose $z \in C(S)$ for some $S \in \mathfrak{X}$, and that we have observed the agent choose x over z (at least once). Would this agent necessarily deem x choosable from $S \cup \{x\}$? This is less clear than the previous situation. The decision maker may have chosen x over z with serious difficulty, perhaps referring to the preferences of another individual. Thus, it is possible that some alternatives in S may dominate x, but not z, with respect to the core preferences of the agent, and this may cause x be not chosen from $S \cup \{x\}$ even though z is deemed choosable from S. This may indeed be the case. However, if z is the *only* choice from S, the following proposition shows that C acts still monotonically with respect to $\mathbf{R}$.

Proposition 4.9. *Let C be a choice correspondence on $\mathfrak{X}$ rationalized by a preference structure $(\succsim, \mathbf{R})$ on X. Then, for any $S \in \mathfrak{X}_{<\infty}$,*

$$x \,\mathbf{R}\, z \text{ and } \{z\} = C(S) \quad \textit{imply} \quad x \in C(S \cup \{x\}).$$

Easy examples would show that the requirement "$\{z\} = C(S)$" cannot be relaxed to "$z \in C(S)$" in Proposition 4.9, nor is this result valid for weak preference structures.

5 Observability of Preference Structures

Rational choice theory is built on the hypothesis that the choice correspondence of a rational individual arises from the maximization of a complete preorder. Moreover, such a choice correspondence is rationalized by a *unique* complete preorder. (That is, if $(X, \mathfrak{X})$ is a choice environment and $\max(\cdot, \succsim) = \max(\cdot, \succsim')$ on $\mathfrak{X}$ for some complete preorders $\succsim$ and $\succsim'$ on X, then $\succsim = \succsim'$.) Thus, every complete preference relation gives rise to a unique rationalizable choice model, and every rationalizable choice model induces a unique preference relation (that arises from pairwise choice problems). While trivial, this duality is an essential aspect of rational choice theory.

In this section, we investigate if, and how, such a duality exists for the choice model induced by preference structures. That is, we examine the relation between two preference structures that happen to rationalize the same choice correspondence. In other words, we would like to understand exactly how two *equivalent* preference structures $(\succsim, \mathbf{R})$ and $(\succsim', \mathbf{R}')$ relate to each other. The exact analogue of the situation in rational choice theory would be to have $\succsim = \succsim'$ and $\mathbf{R} = \mathbf{R}'$ in this instance. The second of these equations is indeed correct (provided that we work with *proper* preference structures), but mainly because different preorders with the same asymmetric part would declare the same elements as maximal in all feasible sets, the first equation is, in general, false. (For instance, $(\triangle_X, X \times X)$ and $(X \times X, X \times X)$ rationalize the same choice correspondence.) However, we will show below that one can always identify "the" largest preference structure – this is the one whose core preference exhibits the least amount of incompleteness – that rationalizes a choice correspondence which is known to be rationalizable by *some* preference structure. Thus, the present choice model too exhibits a useful, and this time entirely nontrivial, duality. Every preference structure gives rise to a unique rationalizable choice model (in the context of a suitably general choice environment), and conversely, every choice model that is rationalizable by a preference structure induces a unique largest preference structure.

Revealed Preferences. Let $(X, \mathfrak{X})$ be any choice environment, and C a choice correspondence on $\mathfrak{X}$. We define the binary relation $\mathbf{R}_C$ on X as

$$x \, \mathbf{R}_C \, y \quad \text{iff} \quad x \in C\{x, y\}.$$

In words, $x \, \mathbf{R}_C \, y$ means that the agent (with choice correspondence C) would choose x over y when comparing these two alternatives alone. Naturally, we refer to $\mathbf{R}_C$ as the *revealed preference relation* induced by C. This relation is complete (because C is nonempty-valued over finite sets).

The following elementary observation highlights the importance of $\mathbf{R}_C$.

Lemma 5.1. *Let $(X, \mathfrak{X})$ be any choice environment, and C a choice correspondence on $\mathfrak{X}$. If C is rationalized by a preference structure $(\succsim, \mathbf{R})$ on X, then $\mathbf{R} = \mathbf{R}_C$.*

Proof. For any $x, y \in X$, setting $S = \{x, y\}$ in (3) yields $\{x\} = C\{x, y\}$ iff $x \, \mathbf{R}^{>} \, y$, and $\{x, y\} = C\{x, y\}$ iff $x \, \mathbf{R}^{=} \, y$. Thus: $\mathbf{R} = \mathbf{R}_C$. ■

In other words, the revealed preferences of an individual whose choices are rationalized by a preference structure are uniquely identified from her binary choice decisions. This shows that the interpretation of preference structures we outlined in Section 3.1 is duly consistent with the choice theory we introduced in Section 4.

We should emphasize that Lemma 5.1 is not valid for *weak* preference structures. Indeed, if $(\succsim, \mathbf{R})$ is weak preference structure on X, then $(\succsim, \mathbf{R}) \cong (\succsim, \mathbf{R}_{\succsim})$ by Proposition 4.1, but, in general, $\mathbf{R}$ and $\mathbf{R}_{\succsim}$ are distinct. Thus, at least from the perspective of preference identification, proper preference structures appear to be superior to weak preference structures.

Revealed Core Preferences. Unlike its revealed part, the core preference part of a preference structure is not observable by an outside observer, so it is particularly important to understand which sorts of preorders on X rationalize a given choice correspondence C on $\mathfrak{X}$ when coupled with $\mathbf{R}_C$. We denote the set of all such preorders by $\mathbb{P}(C)$, that is,

$$\mathbb{P}(C) := \{\succsim \,:\, (\succsim, \mathbf{R}_C) \text{ is a preference structure on } X \text{ that rationalizes } C\}.$$

Thus: $\mathbb{P}(C) \neq \emptyset$ *iff C is rationalizable by a preference structure*. Moreover, for any $\succsim$ and $\succsim'$ in $\mathbb{P}(C)$, we have $(\succsim, \mathbf{R}_C) \cong (\succsim', \mathbf{R}_C)$, that is, choice-theoretically, there is no difference between $\succsim$ and $\succsim'$.

There is a natural way of partially ordering all preorders on X on the basis of their completeness. For any two such preorders $\succsim$ and $\succsim'$, we say that $\succsim$ is **more complete than** $\succsim'$ if the former preorder extends the latter. We denote this partial order (as well as any restriction of it to a given set of preorders on X) by $\sqsupseteq$. In particular, for any $\succsim$ and $\succsim'$ in $\mathbb{P}(C)$, we have

$$\succsim \;\sqsupseteq\; \succsim' \quad \text{iff} \quad \succsim \text{ extends } \succsim'.$$

Clearly, endowing $\mathbb{P}(C)$ with $\sqsupseteq$ makes it a poset. In fact, even at this level of generality, this poset possesses a nontrivial structure:

Theorem 5.2. *Let $(X, \mathfrak{X})$ be any choice environment, and C a choice correspondence on $\mathfrak{X}$. If C is rationalized by at least one preference structure on X, then $\mathbb{P}(C)$ is a complete $\vee$-semilattice.*[12]

This result, whose proof is somewhat involved, may at first appear as a technical observation, but, in fact, it provides a clear insight about those core preferences that rationalize a

[12] A poset $(A, \trianglerighteq)$ is said to be a *complete $\vee$-semilattice* if for every nonempty subset B of A, the $\trianglerighteq$-minimum of the set $\{a \in A : a \trianglerighteq B\}$ exists; *complete $\wedge$-semilattices* are defined dually. If $(A, \trianglerighteq)$ is both a complete $\vee$-semilattice and a complete $\wedge$-semilattice, we say that it is a *complete lattice*.

given choice correspondence C when coupled with the revealed preference relation induced by C. Apparently, any collection of such core preference relations can be combined to get another, more decisive, core preference that still rationalizes C when adjoined to $\mathbf{R}_C$. In particular, there is a *most decisive* core preference on X. (We will compute this preference shortly.)

Remark. $\mathbb{P}(C)$ need not be a $\wedge$-semilattice under the hypotheses of Theorem 5.2. To see this, pick any two objects outside $\mathbb{N}^2$, say, a and b, and put $X := \mathbb{N}^2 \sqcup \{a, b\}$. Next, take the partial order $\succeq_{\mathrm{I}}$ on X with $x \succ_{\mathrm{I}} y$ iff either $x, y \in \mathbb{N}^2$, $x_1 > y_1$ and $x_2 = y_2$, or $x \in \mathbb{N}^2$ and $y = a$. Similarly, let $\succeq_{\mathrm{II}}$ stand for the partial order on X with $x \succ_{\mathrm{II}} y$ iff either $x, y \in \mathbb{N}^2$, $x_1 = y_1$ and $x_2 > y_2$, or $x \in \mathbb{N}^2$ and $y = a$. Finally, define the binary relation $\mathbf{R}$ on X as $x \mathbf{R} y$ iff either $x, y \in \mathbb{N}^2$ and $x_1 + x_2 \geq y_1 + y_2$, or $y \in \{a, b\}$. Then, $(\succeq_{\mathrm{I}}, \mathbf{R})$ and $(\succeq_{\mathrm{II}}, \mathbf{R})$ are transitive preference structures on X. Now, put $\mathfrak{X} := \mathfrak{X}_{<\infty} \cup \{X\}$, and note that $(X, \mathfrak{X})$ is a choice environment. Besides, both $(\succeq_{\mathrm{I}}, \mathbf{R})$ and $(\succeq_{\mathrm{II}}, \mathbf{R})$ rationalize the choice correspondence C on $\mathfrak{X}$, where $C(S) = \max(S, \mathbf{R})$ for any $S \in \mathfrak{X}_{<\infty}$ and $C(X) = \{b\}$. Thus: $\{\succeq_{\mathrm{I}}, \succeq_{\mathrm{II}}\} \subseteq \mathbb{P}(C)$. But, if $\succsim$ is a preorder on X with $\succeq_{\mathrm{I}} \supseteq \succsim$ and $\succeq_{\mathrm{II}} \supseteq \succsim$, then no two elements of $\mathbb{N}^2$ are $\succsim$-comparable, so $\mathbf{MAX}(X, \succsim) \supseteq \mathbb{N}^2$, and hence, $\bigcirc(\mathbf{MAX}(X, \succsim), \mathbf{R}) = \varnothing$, which means that $(\succsim, \mathbf{R})$ does not rationalize C. Thus, there is no infimum of $\succeq_{\mathrm{I}}$ and $\succeq_{\mathrm{II}}$ in $\mathbb{P}(C)$ relative to $\sqsupseteq$.

The Largest Revealed Core Preference. The revealed preference relation induced by C arises only through pairwise comparisons of alternatives, and it does not tell us whether or not $x \mathbf{R}_C y$ entails the choosability of x over y across all feasible sets. This task would be handled by those preorders $\succsim$ that are compatible with $\mathbf{R}_C$. Indeed, according to our interpretation of $(\succsim, \mathbf{R}_C)$, $x \succ y$ means that the agent prefers x over y "obviously," so it stands to reason that she would never choose y in any situation in which x is feasible; the presence of x in any menu rules out y being a potential choice. This prompts looking at the asymmetric binary relation $\succ_C$ on X with

$$x \succ_C y \quad \text{iff} \quad y \notin C(S) \text{ for every } S \in \mathfrak{X} \text{ with } x \in S.$$

We refer to $\succ_C$ as the *revealed core dominance* induced by C.

It is also possible that x and y are "obviously" equally appealing for the decision maker. From the vantage point of choice theory, this means that replacing x with y in any choice problem does not alter the choice behavior of the agent, apart from the replacement of x with y. That is, if x is deemed choosable (or unchoosable) in a menu, replacing x with y in that menu would yield a menu in which y is choosable (or, respectively, unchoosable). In addition, the choosability status of any other alternative in the two menus remains the same. We are thus led to define the binary relation $\sim_C$ on X by $x \sim_C y$ iff

$$x \in C(S \cup \{x\}) \text{ iff } y \in C(S \cup \{y\})$$

and

$$z \in C(S \cup \{x\}) \text{ iff } z \in C(S \cup \{y\})$$

for every $S \in \mathfrak{X}$ and every $z \in S$. Put succinctly, $x \sim_C y$ means that x and y are perfect substitutes in that replacing one for the other does not change the choice behavior of the agent in any choice situation. This relation, which we borrow from Riberio and Riella (2017), is called the *revealed core indifference* induced by C. It is a symmetric relation disjoint from $\succ_C$.

Finally, we define $\succsim_C$ as the union of the relations $\succ_C$ and $\sim_C$, and refer to it as the *revealed core preference relation* induced by C. In general, $\succsim_C$ is a subrelation of $\mathbf{R}_C$ distinct from $\mathbf{R}_C$. In fact, not only is $(\succsim_C, \mathbf{R}_C)$ is a preference structure on X that rationalizes C, but it is the largest such structure. Put differently, $\succsim_C$ turns out to be the top element of the complete $\vee$-semilattice $\mathbb{P}(C)$. This is the second main result of this section:

Theorem 5.3. *Let $(X, \mathfrak{X})$ be any choice environment, and C a nonempty-valued choice correspondence on $\mathfrak{X}$. If C is rationalized by at least one preference structure on X, then*

$$\bigvee \mathbb{P}(C) = \succsim_C .$$

Once again, this is not meant to be a technical result. Rather, it characterizes the most decisive core preference relation compatible with a choice correspondence (that is rationalizable by a preference structure). While the core part of a preference structure is, in general, not observable and non-unique, we can still elicit this part, in its most decisive form, from one's choice behavior.

Taking stock: Given any choice correspondence C that is rationalized by a preference structure, we can identify $(\succsim_C, \mathbf{R}_C)$ as the preference structure of the involved decision-maker. Insofar as the values of C are observable across all choice problems, $(\succsim_C, \mathbf{R}_C)$ is perfectly observable, and further, the choice correspondence that $(\succsim_C, \mathbf{R}_C)$ rationalizes is precisely C. Besides, we can be sure that $\mathbf{R}_C$ corresponds to the actual revealed preference of the agent. While $\succsim_C$ may not be identical to her actual core preference, it is the most decisive version of that preference. Indeed, it is "equivalent" to it insofar as the choices of the agent is concerned. And it is the most complete of all preference relations that is equivalent to her actual core preference in this sense. This identifies in exactly how we may recover one's preference structure from her observed choice behavior. In particular, so long as we pick the revealed core preference and revealed preference relations induced by C as "representative," then the models $(\succsim_C, \mathbf{R}_C)$ and C stand dual to each other.

Example 5.1. (*Revealed Preferences with Imperfect Discrimination*) Take any integer $n \geq 2$, and let $u : \mathbb{R}^n \to \mathbb{R}$ be a continuous surjection. Pick any $\varepsilon > 0$, and consider the preference structure $(\succsim, \mathbf{R})$ on $\mathbb{R}^n$ where $x \mathbf{R} y$ iff $u(x) \geq u(y) - \varepsilon$, and $x \succsim y$ iff either $x = y$ or $u(x) > u(y) + \varepsilon$. Let C be the choice correspondence on $\mathbf{k}(\mathbb{R}^n)$ rationalized by $(\succsim, \mathbf{R})$. Using the characterization of C given in Example 4.5, one can show that

$$x \succsim_C y \quad \text{iff} \quad \text{either } u(x) = u(y) \text{ or } u(x) > u(y) + \varepsilon.$$

So, in this case, we have $\succ = \succ_C$, but (as u cannot be injective), we have $\succsim \neq \succsim_C$.

Remark. For expositional purposes, we have not stated Theorem 5.3 above in its strongest form. It turns out that the nonempty-valuedness hypothesis can be omitted in the statement of this theorem, but at the cost of lengthening the proof significantly. In view of Theorem 4.3, however, this hypothesis is largely inconsequential for applications.

Remark. While it applies in *any* choice environment, Theorem 5.2 has a shortcoming. The ordering of $\mathbb{P}(C)$ used in Theorem 5.2 exhibits a somewhat unnatural asymmetry: While any two rationalizing core preferences can be combined to get a more decisive such preference, we cannot, in general, find a less decisive such preference. It is worth noting that in most applications this issue does not arise. Indeed, if X is a topological space, and C is the choice correspondence on $\mathbf{k}(X)$ rationalized by a continuous preference structure $(\succsim, \mathbf{R})$ on X, then $\mathbb{P}(C)$ is a complete lattice. The proof of this result is avaliable from the authors upon request.[13]

6 Conclusion

In this paper, we proposed a new model to describe the preferences of an economic agent on an arbitrarily given set X of choice prospects. The classical approach is to use a com-

[13] It may be of interest to see what the *least decisive* core preference that rationalizes C when combined with $\mathbf{R}_C$, namely, $\bigwedge \mathbb{P}(C)$ looks like. This relation is found as $\text{tran}(\rhd) \cup \triangle_X$, where $\rhd$ is a binary relation on X defined by $x \rhd y$ iff there exist a $k \in \mathbb{N}$ and $z_1, \dots, z_k$ in X such that (i) $y \mathbf{R}_C z_1 \mathbf{R}_C \cdots \mathbf{R}_C z_k \mathbf{R}_C x$, (ii) $y \in C\{y, z_1, \dots, z_k\}$, (iii) $\{x, z_1, \dots, z_k\} \subseteq C(S)$ for some $S \in \mathbf{k}(X)$ with $y \in S$, and (iv) $y \notin C\{x, y, z_1, \dots, z_k\}$. Again, the proof is available upon request.

plete preorder, which is typically referred to as a *preference relation*, on X for this purpose. Instead, we suggested the use of two binary relations on X. The first of these, denoted as $\succsim$, aims to capture those rankings of the agent that are (subjectively) "obvious/easy." As it is hard to imagine that cyclical choice patterns would arise from the "easy" pairwise choice problems, we assume that $\succsim$ is reflexive and transitive, but it need not be complete (because some pairwise choice problems may well be deemed "hard" by the agent). The second binary relation, denoted as $\mathbf{R}$, arises from what we observe the agent choose in the context of all pairwise choice problems. As these include the "hard" ones as well, this relation may exhibit cycles, so it is allowed to be nontransitive, while, naturally, we assume that it is complete. Finally, we posit that $\succsim$ and $\mathbf{R}$ are consistent with each other (as they arise from the preferences of the same agent) in the sense that (i) $\mathbf{R}$ is an extension of $\succsim$, and (ii) $\mathbf{R}$ is transitive with respect to $\succsim$. This way we arrive at what we dubbed here as a *preference structure*. We have showed above that many preference models (where the decision-making unit may be a group of individuals) are captured by such structures. Among these are the models of incomplete preferences, preferences with imperfect ability of discrimination, regret preferences, and preferences completed by the recommendations of a consultant.

As the main goal of this paper, we have developed a model of choice behavior that arises from preference structures by using the notion of top-cycles. This led to a rich theory of choice which generalizes the classical rational choice theory. The explanatory power of this alternative choice theory is obviously superior to the classical one. It also has a good deal of predictive power (although, of course, less than the classical theory), for it is a menu-independent model that satisfies, say, the Condorcet Criterion. Moreover, this theory has appealing existence and uniqueness properties, paralleling those of the standard rational choice theory. Indeed, one of our main results establishes the nonempty-valuedness of choice correspondences that are induced by preference structures, and another one identifies the largest preference structure that rationalizes a choice correspondence that is known to be rationalizable by some such structure.

We would like to think of the present paper as a beginning of an extensive research project with numerous avenues to be explored. It would be interesting to revisit the classical consumer theory, this time using preference structures instead of preference relations. Similarly, and even more interestingly, one should investigate how (ordinal) game theory would look like when we model the preferences of the players through preference structures. Then, one should certainly see how the classical theories of decision-making under risk and uncertainty would adapt to preference structures. This would, in turn, open up a whole new set of potential applications. Similarly, it should be interesting to see how one may model time preferences through preference structures, and then revisit the theory of optimal saving. These, and numerous other avenues that remain to be explored, will eventually determine if the notion of preference structures is indeed a useful construct for decision theory at large.

APPENDIX: Proofs

This appendix contains the proofs of the results that were omitted in the body of the text.

Proof of Proposition 3.1

Let X be a nonempty set, and $(\succsim, \mathbf{R})$ a weak preference structure on X. That $\mathbf{R}_{\succsim}$ is a completion of $\succsim$ follows readily from the definition of $\mathbf{R}_{\succsim}$, so we need only to prove that $\mathbf{R}$ is $\succsim$-transitive. To this end, take any x, y and z in X such that $x \;\mathbf{R}_{\succsim}\; y \succsim z$. Notice that $z \succ x$ cannot hold, because otherwise $y \succ x$ (by transitivity of $\succsim$), and hence $y \;(\mathbf{R}_{\succsim})^{\succ}\; x$ (because $\mathbf{R}_{\succsim}$ is an extension of $\succsim$), a contradiction. Thus: Either $x \succsim z$ or $(x, z) \in \text{Inc}(\succsim)$. In the former case, we have $x \;\mathbf{R}_{\succsim}\; z$ by definition of $\mathbf{R}_{\succsim}$, so we are done. Similarly, if $x \succsim y$, then $x \;\mathbf{R}_{\succsim}\; z$ because $\succsim$ is transitive and $\succsim \subseteq \mathbf{R}_{\succsim}$. So, assume that $(x, z) \in \text{Inc}(\succsim)$ and $x \succsim y$ is false. Since $x \;\mathbf{R}_{\succsim}\; y$, the latter statement and the definition of $\mathbf{R}_{\succsim}$ imply that $x \;\mathbf{R}\; y$. Then, $x \;\mathbf{R}\; y \succsim z$, and hence $x \;\mathbf{R}\; z$ by $\succsim$-transitivity of $\mathbf{R}$. It follows that $(x, z) \in \text{Inc}(\succsim)$ and $x \;\mathbf{R}\; z$, that is, $x \;\mathbf{R}_{\succsim}\; z$, as we sought. As we can similarly show that $x \succsim y \;\mathbf{R}_{\succsim}\; z$ implies $x \;\mathbf{R}_{\succsim}\; z$, we conclude that $(\succsim, \mathbf{R}_{\succsim})$ is a preference structure on X.

Proof of Corollary 3.2

Let X be a nonempty set, and $(\succsim, \mathbf{R})$ a transitive weak preference structure on X. In view of Proposition 3.1, we only need to prove that $\mathbf{R}_{\succsim}$ is quasitransitive. We will in fact prove something stronger than this below.

First, recall that a binary relation $\mathbf{S}$ is said to be **Suzumura consistent** if x tran($\mathbf{S}$) y implies not $y \;\mathbf{S}^{\succ}\; x$ for every $x, y \in X$. Second, note that it is easy to verify that if $(\succsim, \mathbf{S})$ is a preference structure such that $\text{Inc}(\succsim) \cap \mathbf{S}$ is Suzumura consistent, then $\mathbf{S}$ is quasitransitive. Thus, Proposition 3.1 will be proved if we can show that $\text{Inc}(\succsim) \cap \mathbf{R}_{\succsim}$ is Suzumura consistent. To this end, put $\mathbf{T} := \text{Inc}(\succsim) \cap \text{tran}(\mathbf{R}_{\succsim})$, and note that

$$\mathbf{T} = \text{Inc}(\succsim) \cap \mathbf{R}_{\succsim}. \tag{7}$$

(Indeed, for any $(x, y) \in \mathbf{T}$, there exist finitely many $x_1, ..., x_k \in X$ such that $x \;\mathbf{R}_{\succsim}\; x_1 \;\mathbf{R}_{\succsim} \cdots \mathbf{R}_{\succsim}\; x_k \;\mathbf{R}_{\succsim}\; y$. Since $\mathbf{R}_{\succsim} \subseteq \succsim \cup \mathbf{R} \subseteq \mathbf{R}$, we then have $x \;\mathbf{R}\; x_1 \mathbf{R} \cdots \mathbf{R}\; x_k \;\mathbf{R}\; y$, so given that $\mathbf{R}$ is transitive, we find $x \;\mathbf{R}\; y$. As $(x, y) \in \text{Inc}(\succsim)$, this means $x \;\mathbf{R}_{\succsim}\; y$. Thus, $\mathbf{T} \subseteq \text{Inc}(\succsim) \cap \mathbf{R}_{\succsim}$, while the converse containment is trivially true.) Now, to derive a contradiction, suppose $\text{Inc}(\succsim) \cap \mathbf{R}_{\succsim}$ is not Suzumura consistent. Then, by (7), $\mathbf{T}$ is not Suzumura consistent, so there exists a $k \in \mathbb{N}$ and $x_1, \ldots, x_k$ in X such that $x_1 \,\mathbf{T} \cdots \mathbf{T}\, x_k \,\mathbf{T}\, x_1$ with at least one of these $\mathbf{T}$ holding strictly. By relabeling if necessary, we may assume that $x_k \,\mathbf{T}^{>}\, x_1$. Then, $x_1 \,\mathbf{R}_{\succsim} \cdots \mathbf{R}_{\succsim}\, x_k \,\mathbf{T}^{>}\, x_1$, that is, $x_1 \operatorname{tran}(\mathbf{R}_{\succsim})\, x_k\, (\mathbf{R}_{\succsim})^{>}\, x_1$. Since x_1 and x_k are not $\succsim$-comparable, it follows that $x_1 \,\mathbf{T}\, x_k \,\mathbf{T}^{>}\, x_1$, a contradiction.

Proof of Theorem 3.3

The proof of the "if" part of the assertion is straightforward, so we focus only on its "only if" part. Let $(\succsim, \mathbf{R})$ be a preference structure on X. (The proof for a weak preference structure is given analogously.) Put $\mathbf{T} := \mathbf{R} \setminus \succsim$. We may assume that $\mathbf{T}$ is nonempty, for otherwise there is nothing to prove.

Claim. For every $(x, y) \in \mathbf{T}$, there is a preorder $\succsim_{(x,y)}$ on X such that (i) $\succsim_{(x,y)}$ extends $\succsim$, (ii) $\succsim_{(x,y)} \subseteq \mathbf{R}$, and (iii) $x \succsim_{(x,y)} y$.[14]

Proof of Claim. Fix any $(x, y) \in \mathbf{T}$, and define

$$\succsim_{(x,y)} \;:= \succsim \cup\, (x^{\uparrow,\succsim} \times y^{\downarrow,\succsim}).$$

That $\succsim_{(x,y)}$ is a preorder with $x \succsim_{(x,y)} y$ is verified routinely. To prove (ii), take any $a, b \in X$ with $a \succsim_{(x,y)} b$. If (a, b) does not belong to $\mathbf{R}$, then it does not belong to $\succsim$ either (because $\mathbf{R}$ is a superrelation of $\succsim$). In that case, then, (a, b) belongs to $x^{\uparrow,\succsim} \times y^{\downarrow,\succsim}$, so we have $a \succsim x \,\mathbf{R}\, y \succsim b$, which, by $\succsim$-transitivity of $\mathbf{R}$, implies $a \,\mathbf{R}\, b$, a contradiction. We thus conclude that $\succsim_{(x,y)} \subseteq \mathbf{R}$. It remains to check that $\succsim_{(x,y)}$ extends $\succsim$. Obviously, $\succsim$ is a subrelation of $\succsim_{(x,y)}$. To complete the proof of the claim, then, take any $a, b \in X$ with $a \succ b$. To derive a contradiction, suppose we have $b \succsim_{(x,y)} a$. Then, by definition of $\succsim_{(x,y)}$, (b, a) must belong to $x^{\uparrow,\succsim} \times y^{\downarrow,\succsim}$, and hence, $y \succsim a \succ b \succsim x$. As $\succsim$ is transitive, then, $y \succ x$, and this implies $y \,\mathbf{R}^{>}\, x$ (because $\succ$ is a subrelation of $\mathbf{R}^{>}$), but this contradicts the fact that $(x, y) \in \mathbf{R}$. $\|$

For each $(x, y) \in \mathbf{T}$, let $\succsim_{(x,y)}$ be a preorder on X that satisfies the conditions of the claim above, and put

$$\mathcal{P} := \{\succsim_{(x,y)} : (x, y) \in \mathbf{T}\} \cup \{\succsim\}.$$

Then, every element of $\mathcal{P}$ is a preorder on X that extends $\succsim$. As $\succsim$ is a subrelation of $\succsim_{(x,y)}$, and $\succsim_{(x,y)}$ is a subrelation of $\mathbf{R}$, for each $(x, y) \in \mathbf{T}$, it is also plain that $\succsim = \bigcap \mathcal{P}$ and $\bigcup \mathcal{P} \subseteq \mathbf{R}$. On the other hand, if $x \,\mathbf{R}\, y$, then either $x \succsim y$ or $(x, y) \in \mathbf{T}$. In the former case, we obviously have $(x, y) \in \bigcup \mathcal{P}$. In the latter case, $x \succsim_{(x,y)} y$, and we again find $(x, y) \in \bigcup \mathcal{P}$. Conclusion: $\bigcup \mathcal{P} = \mathbf{R}$. Finally, as $\mathbf{R}$ is total, this finding shows that $\bigcup \mathcal{P}$ is total, and our proof is complete.

Proof of Corollary 3.4

[14] When $(\succsim, \mathbf{R})$ is a weak preference structure, we verify a weaker property of (i), namely, that $\succsim_{(x,y)}$ includes $\succsim$, which trivially follows by the construction of $\succsim_{(x,y)}$.

For the "if" part of the claim, observe that the hypothesis implies $\mathbf{R}$ is a complete preorder which extends (includes) $\succsim$. Obviously, $\succsim$ is a preorder as it is the intersection of the collection $\mathcal{P}$ of preorders. This concludes that $(\succsim, \mathbf{R})$ is a transitive (weak) preference structure on X. For the "only if" part, we readily obtain the required conditions by setting $\mathcal{P} = \{\succsim, \mathbf{R}\}$.

Proof of Proposition 4.1

We will use the following preliminary result to streamline the argument.

Lemma A.1. *Let S be a nonempty set, and $\mathbf{R}$ a complete binary relation on X such that $\bigcirc(S, \mathbf{R}) \neq \emptyset$. Then, x tran$(\mathbf{R}|_S)$ y for every $x, y \in \bigcirc(S, \mathbf{R})$.*

Proof. Suppose the assertion is false. By completeness of $\mathbf{R}$, then, there is a $y \in \bigcirc(S, \mathbf{R})$ such that $A := \{x \in \bigcirc(S, \mathbf{R}) : x \text{ tran}(\mathbf{R}|_S)^{>} y\}$ is nonempty. Then, A tran$(\mathbf{R}|_S)^{>}$ z, which implies $A\ \mathbf{R}|_S^{>}\ z$ (because $\mathbf{R}$ is complete), for every $z \in \bigcirc(S, \mathbf{R}) \backslash A$. But then A is a proper subset of $\bigcirc(S, \mathbf{R})$ which is an $\mathbf{R}$-highset in S, which contradicts $\bigcirc(S, \mathbf{R})$ being the smallest such set. ■

We now turn to the proof of Proposition 4.1. If $\bigcirc(S, \mathbf{R}) \neq \emptyset$, then Lemma A.1, and the fact that $\bigcirc(S, \mathbf{R})$ is an $\mathbf{R}$-highset in S, readily entail that any one element of $\bigcirc(S, \mathbf{R})$ is a maximum element in S with respect to tran$(\mathbf{R}|_S)$. In other words, nonemptiness of $\bigcirc(S, \mathbf{R})$ entails that max(S,tran$(\mathbf{R}|_S)$) is nonempty. Consequently, it is enough to prove the desired equation under the hypothesis that max(S,tran$(\mathbf{R}|_S)$) $\neq \emptyset$. If x belongs to max(S,tran$(\mathbf{R}|_S)$) and y is an element of S that does not, then $y\ \mathbf{R}\ x$ cannot hold, because otherwise, $y\ \mathbf{R}\ x$ tran$(\mathbf{R}|_S)$ S, and hence, y tran$(\mathbf{R}|_S)$ S, which means $y \in$ max(S,tran$(\mathbf{R}|_S)$), a contradiction. As $\mathbf{R}$ is complete and max(S,tran$(\mathbf{R}|_S)$) is nonempty, therefore, we conclude that max(S,tran$(\mathbf{R}|_S)$) is an $\mathbf{R}$-highset in S. To derive a contradiction, suppose there is an $\mathbf{R}$-highset in S, say, B, which is a proper subset of max(S,tran$(\mathbf{R}|_S)$). Take any x in max(S,tran$(\mathbf{R}|_S)$) which does not belong to B, and fix an arbitrary y in B. As x tran$(\mathbf{R}|_S)$ y, there exist finitely many $a_1, ..., a_k \in S$ such that $x\ \mathbf{R}\ a_1\ \mathbf{R} \cdots \mathbf{R}\ a_k\ \mathbf{R}\ y$. Then, since B is an $\mathbf{R}$-highset in S that contains y, it must also contain a_k. Continuing inductively with this argument, we see that each a_i, and in fact, x must belong to B, a contradiction. This completes our proof.

In the remaining part of this appendix, we adopt the following two conventions:

Notational Conventions: In what follows, where a preorder $\succsim$ on a nonempty set X is given (and understood from the context), we write $M(A)$ for $\mathbf{MAX}(A, \succsim)$ for any nonempty $A \subseteq X$. Further, for any nonnegative integer k, we put $[k] := \{0, ..., k\}$.

Proof of Theorem 4.3

We need the following fact for the main part of the argument.

Lemma A.2. *Let $\succsim$ be a continuous preorder on X. Then, for every $S \in \mathbf{k}(X)$ and $x \in S \backslash M(S)$, there exists a $y \in M(S)$ with $y \succ x$.*

Proof. Take any $S \in \mathbf{k}(X)$ and $x \in S \backslash M(S)$, and put $T := \{y \in S : y \succsim x\}$. By upper semicontinuity of $\succsim \cap (S \times S)$, T is a closed subset of S. Since S is compact, therefore, T is a compact set in X. Then, by means of a well-known theorem of order-theory, we have $\mathbf{MAX}(T) \neq \varnothing$.[15] Pick any y in this set. Notice that any $z \in S$ with $z \succsim y$ must belong to T (by transitivity of $\succsim$). It follows that y is $\succsim$-maximal in S as well. And, obviously, $y \succsim x$. Besides, since x is not $\succsim$-maximal in S, we have $y \succ x$. ■

We now turn to the proof of Theorem 4.3. Let $(\succsim, \mathbf{R})$ be a continuous preference structure on X. Take any S in $\mathbf{k}(X)$, and note that $M(S) \neq \varnothing$ (Lemma A.2). If there is an $\mathbf{R}$-maximum element in $M(S)$, then, obviously, this element is $\text{tran}(\mathbf{R}|_{M(S)})$-maximum in $M(S)$, and hence it belongs to $\bigcirc(M(S), \mathbf{R})$. Assume, then, there is no $\mathbf{R}$-maximum in $M(S)$. This means that for every $x \in M(S)$, there is a $y \in M(S)$ with $y \mathbf{R}^{>} x$. Moreover, take an arbitrary $x \in S \backslash M(S)$, and observe that, by Lemma A.2, there exists a $z \in M(S)$ with $z \succ x$. If this x is such that $x \mathbf{R} y$ for all $y \in M(S)$, then we also have $z \mathbf{R} y$ for all $y \in M(S)$ by $\succsim$-transitivity of $\mathbf{R}$, which contradicts the hypothesis that there is no $\mathbf{R}$-maximum in $M(S)$. Therefore, $y \mathbf{R}^{>} x$ for some $y \in M(S)$. Conclusion: For every $x \in S$, there is a $y \in M(S)$ with $y \mathbf{R}^{>} x$. It follows that $\{y^{\downarrow\downarrow} : y \in M(S)\}$ is an open cover of S, where $y^{\downarrow\downarrow} := \{x \in S : y \mathbf{R}^{>} x\}$. Since S is compact, then, there is a finite subset T of $M(S)$ such that $\{y^{\downarrow\downarrow} : y \in T\}$ covers S. As T is finite, there is a $\text{tran}(\mathbf{R}|_{M(S)})$-maximum, say, x^*, in T. But for any $x \in M(S)$, there is a $y \in T$ with $y \mathbf{R}^{>} x$ (since $\{y^{\downarrow\downarrow} : y \in T\}$ covers S), and hence $x^* \text{ tran}(\mathbf{R}|_{M(S)})\, y \mathbf{R}^{>} x$, that is, $x^* \text{ tran}(\mathbf{R}|_{M(S)})\, x$. It follows that x^* is a $\text{tran}(\mathbf{R}|_{M(S)})$-maximum in $M(S)$. By Proposition 4.1, therefore, $x^* \in \bigcirc(M(S), \mathbf{R})$. This completes the proof of Theorem 4.3.[16]

The following result shows that, in the context of a preference structure $(\succsim, \mathbf{R})$, the asymmetric part of $\mathbf{R}$ is transitive relative to the asymmetric part of $\succsim$.

Lemma A.3. *Let $(\succsim, \mathbf{R})$ be a weak preference structure on a nonempty set X. Then,*

$$x \succsim y \mathbf{R}^{>} z \text{ (or } x \mathbf{R}^{>} y \succsim z) \quad \textit{implies} \quad x \mathbf{R}^{>} z$$

for every $x, y, z \in X$.

Proof. Take any $x, y, z \in X$ with $x \succsim y \mathbf{R}^{>} z$ but assume that $x \mathbf{R}^{>} z$ is false. As $\mathbf{R}$ is complete, we then have $z \mathbf{R} x$. So, $z \mathbf{R} x \succsim y$ and we find $z \mathbf{R} y$ contradicting $y \mathbf{R}^{>} z$. The analogous argument shows that $x \mathbf{R}^{>} y \succsim z$ implies $x \mathbf{R}^{>} z$ as well. ■

[15] The earliest reference for this result seems to be Wallace (1945).

[16] This argument shows that continuity can be relaxed to the hypothesis that $x^{\uparrow,\succsim}$ and $x^{\uparrow,\mathbf{R}}$ are closed in X for every $x \in X$.

Proof of Proposition 4.8

Take any $S \in \mathfrak{X}$, and any $x, z \in X$ with $x \succsim z \in C(S)$. We put $T := S \cup \{x\}$; our aim is to show that $x \in C(T)$. Assume first that $x \sim z$ (where $\sim$ is the symmetric part of $\succsim$). In this case, $M(S) \cup \{x\} = M(T)$. So, in view of Proposition 4.1, $z \in C(S)$ implies that z tran$(\mathbf{R}|_{M(S)})$ $M(S)$ while x $\mathbf{R}^{=}$ z (because $\mathbf{R}$ contains $\succsim$). It follows that x tran$(\mathbf{R}|_{M(T)})$ $M(T)$, so, again by Proposition 4.1, $x \in C(T)$.

Assume now that $x \succ z$. In this case x belongs to $M(T)$, but z does not. To derive a contradiction, let us suppose that x does not belong to $C(T)$. By Proposition 4.1, then, there must exist a $y \in M(T)\backslash\{x\}$ such that

$$y \text{ tran}(\mathbf{R}|_{M(T)})^{>} x. \tag{8}$$

Now, since $z \in C(S)$ and $y \in M(T)\backslash\{x\}$, and hence $y \in M(S)$, we have z tran$(\mathbf{R}|_{M(S)})$ y, so there is a positive integer k and $w_0, ..., w_k \in M(S)$ such that $z = w_0 \ \mathbf{R} \ w_1 \ \mathbf{R} \cdots \mathbf{R} \ w_k = y$. Put $\ell := \max\{i \in [k] : x \succ w_i\}$, which is well-defined because $x \succ w_0$. By (8), and because $\mathbf{R}$ contains $\succsim$, we cannot have $x \succsim w_k$, and hence $\ell \in [k-1]$. But then $w_{\ell+1}, ..., w_k \in M(T)$, and we have $x \succ w_\ell \ \mathbf{R} \ w_{\ell+1} \ \mathbf{R} \cdots \mathbf{R} \ w_k = y$, so, by $\succsim$-transitivity of $\mathbf{R}$, $x \ \mathbf{R} \ w_{\ell+1} \ \mathbf{R} \cdots \mathbf{R} \ w_k = y$. This means x tran$(\mathbf{R}|_{M(T)})$ y, contradicting (8).

Proof of Proposition 4.9

Take any finite $S \in \mathfrak{X}$, and any $x, z \in X$ with $x \ \mathbf{R} \ z$ and $\{z\} = C(S)$. Put $T := S \cup \{x\}$; we wish to show that $x \in C(T)$. Suppose first that x is not $\succsim$-maximal in T. Then $y \succ x$ for some $y \in T$. Since T is finite and $\succsim$ is transitive, it is without loss of generality to assume that $y \in M(T)$. Since $y \succ x \ \mathbf{R} \ z$, we get $y \ \mathbf{R} \ z$ by $\succsim$-transitivity of $\mathbf{R}$. As $\{z\}$ is the top-cycle in $M(S)$, and $y \in M(S)$, we must then have $y = z$. But this means $z \succ x$, and hence $z \ \mathbf{R}^{>} \ x$, contradiction. Conclusion: $x \in M(T)$.

Now, since $x \ \mathbf{R} \ z$, and $\mathbf{R}$ extends $\succsim$, we do not have $z \succ x$. On the other hand, by Proposition 4.8, $x \succsim z$ implies $x \in C(T)$. It remains to consider the case where $(x, z) \in \text{Inc}(\succsim)$. In this case, $z \in M(T)$. Moreover, as $\{z\}$ is the top-cycle in $M(S)$, we have $z \ \mathbf{R}^{>} \ y$ for every $y \in M(S)\backslash\{z\}$. But then, $x \ \mathbf{R} \ z \ \mathbf{R}^{>} \ y$, and hence x tran$(\mathbf{R}|_{M(T)})$ y, for every $y \in M(T)\backslash\{x, z\}$. By Proposition 4.1, then, $x \in C(T)$, and we are done.

Proof of Theorem 5.2

We begin with proving a preliminary result that will be needed in the main body of the proof. This lemma is stated in the setting of Theorem 5.2.

Lemma A.4. *For any $S \in \mathfrak{X}$, $\succsim \in \mathbb{P}(C)$, and any $(x, y) \in S \times X$ with $x \succsim y$,*

$$x \succsim y \quad \textit{implies} \quad C(S \cup \{y\}) \cap S = C(S).$$

Proof. If y is not $\succsim$-maximal in $S \cup \{y\}$, then $M(S \cup \{y\}) = M(S)$, and the claim follows readily from Proposition 4.1. We thus assume that $y \in M(S \cup \{y\})$. In turn, since $x \succsim y$, this implies that $x \in \mathbf{MAX}(S \cup \{y\})$ and $x \sim y$. Consequently,

$$\{x, y\} \subseteq M(S \cup \{y\}) = M(S) \cup \{y\}. \tag{9}$$

Besides, for any $a, b \in M(S)$, we have

$$a \operatorname{tran}(\mathbf{R}|_{M(S)})\, b \quad \text{iff} \quad a \operatorname{tran}(\mathbf{R}|_{M(S)\cup\{y\}})\, b, \tag{10}$$

where we denote $\mathbf{R}_C$ by $\mathbf{R}$ to simplify the notation. (The "only if" part of (10) is trivial. Its "if" part follows from the fact that $z\,\mathbf{R}\,y$ implies $z\,\mathbf{R}\,x$, and $y\,\mathbf{R}\,z$ implies $x\,\mathbf{R}\,z$, for any $z \in S$ (because $x \sim y$, and $\mathbf{R}$ is $\succsim$-transitive).) Now, there are three cases to consider.

Case 1. $C(S) = \varnothing$. In this case, by Proposition 4.1, there is no $\operatorname{tran}(\mathbf{R}|_{M(S)})$-maximum in $M(S)$. It then follows from (10) that there is no $\operatorname{tran}(\mathbf{R}|_{M(S)\cup\{y\}})$-maximum in $M(S) \cup \{y\}$. Since $M(S \cup \{y\}) = M(S) \cup \{y\}$, then, Proposition 4.1 entails $C(S \cup \{y\}) \cap S = \varnothing$.

Case 2. $C(S) \neq \varnothing$ and $x \in S \backslash C(S)$. In this case, we have $C(S)\, \mathbf{R}^{>}\, x \sim y$. So, by Lemma A.3, $C(S)\, \mathbf{R}^{>}\, y$, that is, $C(S)$ is an $\mathbf{R}$-highset in $M(S) \cup \{y\}$. As $C(S)$ is obviously an $\mathbf{R}$-cycle and (9) holds, we conclude, by Corollary 4.2, that $C(S)$ is the top-cycle in $M(S \cup \{y\})$ with respect to $\mathbf{R}$, that is, $C(S) = C(S \cup \{y\})$.

Case 3. $x \in C(S)$. Again, obviously, $C(S)$ is an $\mathbf{R}$-cycle. Since $x \sim y$ and $\mathbf{R}$ extends $\succsim$, therefore, $C(S) \cup \{y\}$ is an $\mathbf{R}$-cycle as well. Moreover, $y \sim x\, \mathbf{R}^{>}\, M(S) \backslash C(S)$ and hence, $y\, \mathbf{R}^{>}\, M(S) \backslash C(S)$ by Lemma A.3. It follows that $C(S) \cup \{y\}$ is an $\mathbf{R}$-highset in $M(S) \cup \{y\}$. In view of (9) and Corollary 4.2, then, $C(S) \cup \{y\}$ is the top-cycle in $M(S \cup \{y\})$ with respect to $\mathbf{R}$, and hence, $C(S) \cup \{y\} = C(S \cup \{y\})$. ■

We now turn to the proof of Theorem 5.2 in which $\mathcal{P}$ stands for an arbitrarily fixed nonempty subset of $\mathbb{P}(C)$. We define

$$\trianglerighteq_{\mathcal{P}} := \operatorname{tran}\left(\bigcup \mathcal{P}\right),$$

and write $\triangleright_{\mathcal{P}}$ for the asymmetric part of this preorder. (We wish to show that $\trianglerighteq_{\mathcal{P}}$ is the supremum of $\mathcal{P}$ in $\mathbb{P}(C)$ relative to the partial order $\sqsupseteq$.) We organize our main argument in terms of several claims.

Claim 1. $\trianglerighteq_{\mathcal{P}}$ extends any member of $\mathcal{P}$.

Proof of Claim 1. Take any $\succsim$ in $\mathcal{P}$. Obviously, $\trianglerighteq_{\mathcal{P}}$ contains $\succsim$. Next, take any $x, y \in X$ with $x \succ y$. To derive a contradiction, suppose $x \triangleright_{\mathcal{P}} y$ does not hold. Since $x \trianglerighteq_{\mathcal{P}} y$, this means that $y \trianglerighteq_{\mathcal{P}} x$ holds as well. By definition of $\trianglerighteq_{\mathcal{P}}$, then, there exist a $k \in \mathbb{N}$, $\succsim_1, ..., \succsim_k \in \mathbb{P}(C)$ and $z_0, ..., z_k \in X$ such that $x \succ y = z_0 \succsim_1 z_1 \succsim_2 \cdots \succsim_k z_k = x$. Put $S := \{z_0, ..., z_k\}$. Since $C(S) \neq \varnothing$, there is an $i \in [k]$ such that

$z_i \in C(S)$. If $i > 0$, Proposition 4.8 entails that $z_{i-1} \in C(S)$, and continuing inductively, we find that $y = z_0 \in C(S)$. So, in all contingencies, we have $y \in C(S)$. But this is impossible, because y is not $\succsim$-maximal in S, and $(\succsim, \mathbf{R}_C)$ rationalizes C. □

Claim 2. $(\trianglerighteq_{\mathcal{P}}, \mathbf{R}_C)$ is a preference structure on X.

Proof of Claim 2. Let us first show that $\mathbf{R}_C$ extends $\trianglerighteq_{\mathcal{P}}$. Take any $x, y \in X$ with $x \trianglerighteq_{\mathcal{P}} y$. Then, there exist a $k \in \mathbb{N}$, $\succsim_1, ..., \succsim_k \in \mathbb{P}(C)$ and $z_0, ..., z_k \in X$ such that

$$x = z_0 \succsim_1 z_1 \succsim_2 \cdots \succsim_k z_k = y. \tag{11}$$

If $k = 1$, then $x \succsim_1 y$, and hence $x \, \mathbf{R}_C \, y$ (because $\mathbf{R}_C$ is a superrelation of $\succsim_1$). Suppose $k \geq 2$. Then, $z_{k-2} \succsim_{k-1} z_{k-1} \succsim_k z_k$, and hence $z_{k-2} \succsim_{k-1} z_{k-1} \, \mathbf{R}_C \, z_k$, so by $\succsim_{k-1}$-transitivity of $\mathbf{R}_C$, we find

$$x = z_0 \succsim_1 z_1 \succsim_2 \cdots \succsim_{k-2} z_{k-2} \mathbf{R}_C \, z_k = y.$$

If $k = 2$, we are done. Otherwise, we continue this way inductively to obtain $x \, \mathbf{R}_C \, y$ in $k - 1$ steps. Conclusion: $x \, \mathbf{R}_C \, y$. Now assume that we in fact had $x \vartriangleright_{\mathcal{P}} y$ (which implies that at least one of the orderings in (11) holds strictly). Let us show that $x \, \mathbf{R}_C^{=} \, y$ could not hold in this case. Indeed, to derive a contradiction, suppose $x \, \mathbf{R}_C^{=} \, y$ so that $y \in C\{x, y\}$. Then, since $x \succsim_1 z_1$, Lemma A.4 entails that $C\{x, y\} \subseteq C\{x, y, z_1\}$, so $y \in C\{x, y, z_1\}$. If $k \geq 2$, then, since $z_1 \succsim_1 z_2$, Lemma A.4 entails that $y \in C\{x, y, z_1\} \subseteq C\{x, y, z_1, z_2\}$. Continuing this way inductively, we find that $y \in C\{x, y, z_1, ..., z_{k-1}\}$. But then, since $z_{k-1} \succsim_k y$ and $(\succsim_k, \mathbf{R}_C)$ rationalizes C, Proposition 4.8 yields $z_{k-1} \in C\{x, y, z_1, ..., z_{k-1}\}$. Continuing this way inductively, therefore, we find $\{x, y, z_1, ..., z_{k-1}\} = C\{x, y, z_1, ..., z_{k-1}\}$, but this contradicts the fact that at least one of the orderings in (11) holds strictly. Conclusion: $x \vartriangleright_{\mathcal{P}} y$ implies $x \, \mathbf{R}_C^{>} \, y$.

It remains to show that $\mathbf{R}_C$ is $\trianglerighteq_{\mathcal{P}}$-transitive. To this end, take any $x, y, z \in X$ with $x \trianglerighteq_{\mathcal{P}} y \, \mathbf{R}_C \, z$. Then, there are $k \in \mathbb{N}$, $\succsim_1, ..., \succsim_k \in \mathbb{P}(C)$ and $w_0, ..., w_k \in X$ such that $x = w_0 \succsim_1 w_1 \succsim_2 \cdots \succsim_k w_k = y \, \mathbf{R}_C \, z$. So, repeating the induction argument we gave in the previous paragraph, we find $x \, \mathbf{R}_C \, z$. That $x \, \mathbf{R}_C \, y \trianglerighteq_{\mathcal{P}} z$ implies $x \, \mathbf{R}_C \, z$ is similarly proved. □

Claim 3. For any $S \in \mathfrak{X}$ and $\succsim \in \mathcal{P}$,

$$C(S) \subseteq \mathbf{MAX}(S, \trianglerighteq_{\mathcal{P}}) \subseteq \mathbf{MAX}(S, \succsim).$$

Proof of Claim 3. The second containment is an immediate consequence of Claim 1. To establish the first containment, take any x in $C(S)$, and suppose that x is not $\trianglerighteq_{\mathcal{P}}$-maximal in S. Then, there is a $y \in S$ with $y \vartriangleright_{\mathcal{P}} x$, and hence, $y = z_0 \succsim_1 z_1 \succsim_2 \cdots \succsim_k z_k = x$ for some $k \in \mathbb{N}$, $\succsim_1, ..., \succsim_k \in \mathbb{P}(C)$ and $z_0, ..., z_k \in X$, with at least one of these ordering holding strictly. (We have $k > 1$, for otherwise $y \succ_1 x$, that is, x is not $\succsim_1$-maximal in S, contradicting $x \in C(S)$.) Since $y \in S$ and $y \succsim_1 z_1$, Lemma A.4 tells us that $C(S \cup \{z_1\}) \cap S = C(S)$. But then, since $z_1 \in S \cup \{z_1\}$ and $z_1 \succsim_2 z_2$, applying Lemma A.4 again yields

$$C(S \cup \{z_1, z_2\}) \cap (S \cup \{z_1\}) = C(S \cup \{z_1\}).$$

Intersecting both sides of this equation with S, and using the previous equation, then, we find

$$C(S \cup \{z_1, z_2\}) \cap S = C(S).$$

In fact, proceeding this way inductively, we may conclude that

$$C(S \cup \{z_1, ..., z_k\}) \cap S = C(S). \tag{12}$$

In particular, x belongs to $C(S \cup \{z_1, ..., z_k\})$. But then, by Proposition 4.8, z_{k-1} belongs to $C(S \cup \{z_1, ..., z_k\})$ as well. In fact, applying Proposition 4.8 this way inductively, we find that $z_0, ..., z_k \in C(S \cup \{z_1, ..., z_k\})$. But this is impossible, for $z_{i-1} \succ_i z_i$ holds for at least one $i \in \{1, ..., k\}$, so, being not $\succsim_i$-maximal in $S \cup \{z_1, ..., z_k\}$, z_i cannot belong to $C(S \cup \{z_1, ..., z_k\})$. □

Claim 4. $(\trianglerighteq_{\mathcal{P}}, \mathbf{R}_C)$ rationalizes C.

Proof of Claim 4. Take any $S \in \mathfrak{X}$ and note that $C(S)$ is an $\mathbf{R}_C$-cycle in $C(S)$. So, since, by Claim 3, $C(S) \subseteq \mathbf{MAX}(S, \trianglerighteq_{\mathcal{P}})$, it is plain that $C(S)$ is an $\mathbf{R}_C$-cycle in $\mathbf{MAX}(S, \trianglerighteq_{\mathcal{P}})$. Now pick any $\succsim \in \mathbb{P}(C)$. Then, $C(S)$ is an $\mathbf{R}_C$-highset in $\mathbf{MAX}(S, \succsim)$. By Claim 3, therefore, $C(S)$ is an $\mathbf{R}_C$-highset in $\mathbf{MAX}(S, \trianglerighteq_{\mathcal{P}})$ as well. This means that $C(S)$ is the top-cycle in $\mathbf{MAX}(S, \trianglerighteq_{\mathcal{P}})$ with respect to $\mathbf{R}_C$, as we claimed. □

Claims 2 and 4 jointly imply that $\trianglerighteq_{\mathcal{P}}$ belongs to $\mathbb{P}(C)$. It then follows from Claim 1 that $\trianglerighteq_{\mathcal{P}}$ is the supremum of $\mathcal{P}$ in $\mathbb{P}(C)$ relative to $\sqsupseteq$. In view of the arbitrary choice of $\mathcal{P}$ above, we conclude that $\mathbb{P}(C)$ is a complete $\vee$-semilattice relative to this partial order.

Proof of Theorem 5.3

Throughout the proof, we will denote $\mathbf{R}_C$ by $\mathbf{R}$ to simplify the notation. (That is, for any x and y in X, we have $\{x\} = C\{x, y\}$ iff $x \mathbf{R}^{>} y$, and $\{x, y\} = C\{x, y\}$ iff $x \mathbf{R}^{=} y$.) Consequently, $\succ_C \subseteq \mathbf{R}^{>}$ and $\sim_C \subseteq \mathbf{R}^{=}$, that is, that $\mathbf{R}$ extends $\succsim_C$. We will use these facts below as a matter of routine. Also, when $\succsim \in \mathbb{P}(C)$, we again write $M(S) = \mathbf{MAX}(S, \succsim)$ for any $S \in \mathfrak{X}$. The following lemmata are stated in the setting of Theorem 5.3.

Lemma A.5. *For any finite* $S \in \mathfrak{X}$,

$$\{x\} = C(S) \quad \textit{implies} \quad \{x\} = C\{x, y\} \textit{ for every } y \in S.$$

Proof. Take any finite $S \in \mathfrak{X}$ with $\{x\} = C(S)$. If $S = \{x\}$, there is nothing to prove, so assume otherwise, and pick any $y \in S \setminus \{x\}$. Take any $\succsim \in \mathbb{P}(C)$, and denote $\mathbf{R}_C$ by $\mathbf{R}$ to simplify the notation. If $y \in M(S)$, then since $\{x\}$ is the top-cycle in $M(S)$ with respect to $\mathbf{R}$, we have $x\ \mathbf{R}^{>}\ y$. If y is not $\succsim$-maximal in S, then $z \succ y$ holds for some $z \in S$. As S is finite, we may assume that $z \in M(S)$. If $z = x$, then, obviously, $\{x\} = C\{x, y\}$, so assume that $z \neq x$. Then, since $\{x\} = \bigcirc(M(S), \mathbf{R})$, we have $x\ \mathbf{R}^{>}\ z \succ y$, so, by Lemma A.3, we again find $x\ \mathbf{R}^{>}\ y$. Since $\{x\} = C\{x, y\}$ iff $x\ \mathbf{R}^{>}\ y$, we are done. ■

Lemma A.6. *For any $S \in \mathfrak{X}$ and $x \in S$,*

$$x \succ_C y \quad \textit{implies} \quad C(S \cup \{y\}) = C(S).$$

Proof. Take any $S \in \mathfrak{X}$. Let x and y be two elements of X with $x \succ_C y$. Then, $x\ \mathbf{R}^{>}\ y$, so $y \succsim x$ cannot hold (because $\mathbf{R}$ extends $\succsim$). Besides, if there is a $z \in S$ with $z \succ y$, then $C(S \cup \{y\}) \subseteq S$, so, by Lemma A.4, $C(S \cup \{y\}) = C(S)$, and we are done. It remains to consider the case where $(x, y) \in \text{Inc}(\succsim)$ and $y \in M(S \cup \{y\})$.

Since $x \succ_C y$ implies that y does not belong to $C(S \cup \{y\})$, we have

$$C(S \cup \{y\}) \subseteq M(S \cup \{y\}) \cap S \subseteq M(S).$$

Now, we claim that $C(S \cup \{y\})$ is an $\mathbf{R}$-highset in $M(S)$ To see this, take any $\succsim$-maximal z in S that does not belong to $C(S \cup \{y\})$. If $z \in M(S \cup \{y\})$, then we clearly have $C(S \cup \{y\})\ \mathbf{R}^{>}\ z$ (because $C(S \cup \{y\})$ is an $\mathbf{R}$-highset in $M(S \cup \{y\})$. If $z \notin M(S \cup \{y\})$, then, since $y \in M(S \cup \{y\}) \setminus C(S \cup \{y\})$, we have $C(S \cup \{y\})\ \mathbf{R}^{>}\ y \succ z$, which implies $C(S \cup \{y\})\ \mathbf{R}^{>}\ z$ by Lemma A.3. So, $C(S \cup \{y\})$ is an $\mathbf{R}$-highset in $M(S)$. As $C(S \cup \{y\})$ is obviously an $\mathbf{R}$-cycle, $C(S \cup \{y\}) = C(S)$ by Corollary 4.2. ■

We now turn to the proof of Theorem 5.3.

Claim 1. $\mathbf{R}$ is $\succsim_C$-transitive.

Proof of Claim 1. Let us first show that $\mathbf{R}$ is $\succ_C$-transitive. Take any $x, y, z \in X$ with $x\ \mathbf{R}\ y \succ_C z$. If $x\ \mathbf{R}\ z$ is false, then $z\ \mathbf{R}^{>}\ x$ (because $\mathbf{R}$ is complete). Take any $\succsim$ in $\mathbb{P}(C)$. Clearly, $y \succ x$ cannot hold (because $\succsim \subseteq \mathbf{R}$). If, on the other hand, $x \succ y$, then $\{x\} = C\{x, y, z\}$, and by Lemma A.5, this implies $x\ \mathbf{R}^{>}\ z$, a contradiction. Thus: $(x, y) \in \text{Inc}(\succ)$. Similarly, $x \succ z$ cannot hold (because $\succsim \subseteq \mathbf{R}$), and if $z \succ x$, then $\{y\} = C\{x, y, z\}$, and by Lemma A.5, this implies $y\ \mathbf{R}^{>}\ x$, a contradiction. Thus: $(x, y) \in \text{Inc}(\succ)$. Finally, note that $z \succ y$ cannot hold (because $\succ \subseteq \succ_C$), and if $y \succ z$, then Lemma A.3 implies $y\ \mathbf{R}^{>}\ x$, a contradiction. Thus: $(y, z) \in \text{Inc}(\succ)$. Conclusion: $\mathbf{MAX}(\{x, y, z\}, \succsim) = \{x, y, z\}$. Then, by Proposition 4.1, $\{x, y, z\} = C\{x, y, z\}$, but this contradicts $y \succ_C z$. Thus: $\mathbf{R} \circ \succ_C \subseteq \mathbf{R}$. One can similarly prove that $\succ_C \circ \mathbf{R} \subseteq \mathbf{R}$.

We next show that $\mathbf{R}$ is $\sim_C$-transitive. Take any $x, y, z \in X$ with $x\,\mathbf{R}\,y \sim_C z$. The second part of this statement entails that $x \in C(\{x\} \cup \{y\})$ iff $x \in C(\{x\} \cup \{z\})$. But $x \in C\{x, y\}$ (because $x\ \mathbf{R}\ y$), so we find that $x \in C\{x, z\}$, that is, $x \mathbf{R} z$. We thus conclude that $\mathbf{R} \circ \sim_C \subseteq \mathbf{R}$. One can similarly prove that $\sim_C \circ \mathbf{R} \subseteq \mathbf{R}$. □

Claim 2. $\succsim_C$ is transitive.

Proof of Claim 2. Observe that it is sufficient to verify (i) $\sim_C$ is transitive, (ii) $\succ_C$ is transitive, and (iii) $\succ_C$ is $\sim_C$-transitive. The construction of $\sim_C$ readily implies (i). For (ii), suppose that $x \succ_C y \succ_C z$ but $x \succ_C z$ does not hold for some $x, y, z \in X$. Then, there exists an $S \in \mathfrak{X}$ with $x \in S$ and $z \in C(S)$. As $x \succ_C y$, Lemma A.6 implies that $z \in C(S) = C(S \cup \{y\})$. But this is a contradiction since $y \succ_C z$. Thus, $x \succ_C y \succ_C z$ must imply $x \succ_C z$. For (iii), take any $x, y, z \in X$ such that $x \succ_C y \sim_C z$. Let S be an arbitrary element of $\mathfrak{X}$ with $x \in S$. (We wish to show that z does not belong to $C(S)$.) If z does not belong to S, there is nothing to prove, so suppose $z \in S$. As $x \succ_C y$ and $x \in S$, we have $y \notin C(S \cup \{y\})$. Thus, since $y \sim_C z$, we have $z \notin C(S \cup \{z\}) = C(S)$. An analogous argument shows that $x \sim_C y \succ_C z$ implies $x \succ_C z$ as well. □

Claim 3. For any $\succsim \in \mathbb{P}(C)$, $\succsim_C$ extends $\succsim$.

Proof of Claim 3. Let $\succsim \in \mathbb{P}(C)$. If $x \succ y$ for some $x, y \in X$, then $y \notin M(S)$ and thus $y \notin C(S)$ for all $S \in \mathfrak{X}$ with $x \in S$. So, $\succ \subseteq \succ_C$. In the rest of the proof, we prove that $\sim \subseteq \sim_C$. Take any $x, y \in X$ with $x \sim y$. Let S be an arbitrarily fixed element of $\mathfrak{X}$, and put $T_x := M(S \cup \{x\})$ and $T_y := M(S \cup \{y\})$. Since $\succsim$ is a preorder, it is readily checked that $x \sim y$ implies $T := T_x \cap S = T_y \cap S$.

Now assume $x \in C(S \cup \{x\})$. Then, $x \in M(S \cup \{x\})$, and hence, $y \in M(S \cup \{y\})$ (because $\succsim$ is a preorder and $x \sim y$), that is, $y \in T_y$. Moreover, by Proposition 4.1, x $\text{tran}(\mathbf{R}|_{T_x})$ T_x. So, if $z \in T \subseteq T_x$, there is a positive integer k such that $x \mathbf{R} w_0 \mathbf{R} \cdots \mathbf{R} w_k = z$ for some $w_0, ..., w_k \in T_x$. Here, we can in fact assume that $w_0, ..., w_k \in T$ without loss of generality. (For, otherwise, $w_i = x$ for some $i \in [k]$. Then, set $l := \max\{i \in [k] : w_i = x\}$, and we have $x \mathbf{R} w_{l+1} \mathbf{R} \cdots \mathbf{R} w_k = z$ with $w_i \in T$ for all $i = l + 1, \ldots, k$.) Since $y \sim x$ and $\mathbf{R}$ is $\succsim$-transitive, this implies $y \mathbf{R} w_0 \mathbf{R} \cdots \mathbf{R} w_k = z$. Thus, y $\text{tran}(\mathbf{R}|_{T_y})$ T. If $z \in T_y \backslash T$, then $z = y$, and we obviously have y $\text{tran}(\mathbf{R}|_{T_y})$ z. Conclusion: y $\text{tran}(\mathbf{R}|_{T_y})$ T_y. By Proposition 4.1, this yields $y \in C(S \cup \{y\})$, as we sought. By symmetry, therefore, we conclude: $x \in C(S \cup \{x\})$ iff $y \in C(S \cup \{y\})$.

Next, take any z in S with $z \in C(S \cup \{x\})$. Then, $z \in M(S \cup \{x\})$, and hence, $z \in M(S \cup \{y\})$ (because $\succsim$ is a preorder and $x \sim y$), that is, $z \in T_y$. Moreover, by Proposition 4.1, z $\text{tran}(\mathbf{R}|_{T_x})$ T_x. Now, take any $w \in T_y$. Then, we can show that there is a positive integer k with

$$z \mathbf{R} w_0 \mathbf{R} w_1 \mathbf{R} \cdots \mathbf{R} w_k \mathbf{R} w \quad \text{for some } w_0, \ldots, w_k \in T_x. \tag{13}$$

(Indeed, if $w \in T \subseteq T_x$, then z $\text{tran}(\mathbf{R}|_{T_x})$ w, and (13) follows at once. If $w \in T_y \setminus T$, then $y = w \in T_y$, which implies $x \in T_x$ and hence z $\text{tran}(\mathbf{R}|_{T_x})$ $x \sim y$. So, there is a positive interger k such that $z \mathbf{R} w_0 \mathbf{R} w_1 \mathbf{R} \cdots \mathbf{R} w_k \mathbf{R} x \sim y$ for some $w_0, \ldots, w_k \in T_x$. This again implies (13) by $\succsim$-transitivity of $\mathbf{R}$.) For the sequence $w_0, ..., w_k$ in (13), define

$$w_i' := \begin{cases} w_i, & \text{if } w_i \neq x \\ y, & \text{if } w_i = x, \end{cases}$$

for each $i \in [k]$, and note that $z \mathbf{R} w_1' \mathbf{R} \cdots \mathbf{R} w_k' \mathbf{R} w$ by $\gtrsim$-transitivity of $\mathbf{R}$. Since $w_i' \in T_y$ for each $i \in [k]$, this shows that z tran$(\mathbf{R}|_{T_y})$ w. It then follows from the arbitrary choice of w that z tran$(\mathbf{R}|_{T_y})$ T_y, that is, $z \in C(S \cup \{y\})$, as we sought. By symmetry, therefore, we conclude: $z \in C(S \cup \{x\})$ iff $z \in C(S \cup \{y\})$ for every $z \in S$. In view of the arbitrariness of S, this establishes that $\sim \subseteq \sim_C$. □

In view of Theorem 5.2, we have $\bigvee \mathbb{P}(C) \in \mathbb{P}(C)$. So, by Claim 3, it follows that $\bigvee \mathbb{P}(C) \subseteq \gtrsim_C$. As $\mathbf{R}$ extends $\gtrsim_C$, Claim 1 and Claim 2 imply that $(\gtrsim_C, \mathbf{R})$ is a preference structure on X. Since $\bigvee \mathbb{P}(C)$ is the largest preorder in $\mathbb{P}(C)$, if $(\gtrsim_C, \mathbf{R})$ rationalizes C, then $\gtrsim_C \subseteq \bigvee \mathbb{P}(C)$. The next claim estabilishes this step, hence completing the proof of Theorem 5.3.

Claim 4. $(\gtrsim_C, \mathbf{R})$ rationalizes C.

Proof of Claim 4. Take any S in $\mathfrak{X}$ and $\gtrsim \in \mathbb{P}(C)$. If $x \in C(S)$, then $y \succ_C x$ holds for no $y \in S$ by definition of $\succ_C$, implying that $x \in \mathbf{MAX}(S, \gtrsim_C)$. So, $C(S) \subseteq \mathbf{MAX}(S, \gtrsim_C)$. In addition, we have $\mathbf{MAX}(S, \gtrsim_C) \subseteq M(S)$ as $\gtrsim_C$ extends $\gtrsim$ by Claim 3. These observations readily imply that $C(S)$ is an $\mathbf{R}$-highset in $\mathbf{MAX}(S, \gtrsim_C)$. (Indeed, if $x \in C(S)$ and $y \in \mathbf{MAX}(S, \gtrsim_C) \backslash C(S)$, then $y \in M(C) \backslash C(S)$ and thus $x \mathbf{R}^{>} y$.) As $C(S)$ is obviously an $\mathbf{R}$-cycle, we conclude that $C(S) = \bigcirc(\mathbf{MAX}(S, \gtrsim_C), \mathbf{R})$ by Corollary 4.2. The proof is complete. □

References

Armstrong, W., The determinateness of the utility function, *Econ. J.* 49 (1939), 453-467.

Au, P. and K. Kawai, Sequentially rationalizable choice with transitive rationales, *Games Econ. Behav.* 73 (2011), 608-614.

Beja, A. and I. Gilboa, Numerical representations of imperfectly ordered preferences (A unified geometric exposition), *J. Math. Psych.* 36 (1992), 426-449.

Bordalo, P., N. Gennaioli, and A. Schleifer, Salience theory of choice under risk, *Q. J. Econ.* 127 (2012), 1243-1285.

Bewley, T., Knightian uncertainty theory: part I, Cowles Foundation Discussion Paper No. 807 (1986).

Cerreia-Vioglio, S., D. Dillenberger, and P. Ortoleva, Cautious expected utility and the certainty effect, *Econometrica* 83 (2015), 693-728.

Cerreia-Vioglio, S., A. Giarlotta, S. Greco, F. Maccheroni, and M. Marinacci, Rational preference and rationalizable choice, forthcoming in *Econ. Theory*.

Cherepanov, V., T. Feddersen, and A. Sandroni, Rationalization, *Theoret. Econ.* 8 (2013), 775–800.

Costa-Gomez, M., C. Cueva, G. Gerasimou, and M. Tejiscak, Choice, deferral and consistency, *Quantitative Econ.* 13 (2022), 1297-1318.

Costa, M., P. Ramos, and G. Riella, Single-crossing choice correspondences, *Soc. Choice Welfare* 54 (2020).

Danan, E., Revealed preference and indifferent selection, *Math. Soc. Sci.* 55 (2008), 24-37.

Day, B., and G. Loomes, Conflicting violations of transitivity and where they may lead us, *Theory Decis.* 68 (2010), 233-245.

Dubra, J., F. Maccheroni, and E. A. Ok, Expected utility theory without the completeness axiom, *J. Econ. Theory* 115 (2004), 118-133.

Duggan, J., A systematic approach to the construction of non-empty choice sets, *Soc. Choice Welfare* 28 (2007), 491-506.

Ehlers, L., and Y. Sprumont, Weakened WARP and top-cycle choice rules, *J. Math. Econ.* 44 (2008), 87-94.

Eliaz, K. and E. A. Ok, Indifference or indecisiveness: Choice-theoretic foundations of incomplete preferences, *Games Econ. Behav.* 56 (2006), 61-86.

Evren, O., Scalarization methods and expected multi-utility representations, *J. Econ. Theory* 151 (2014), 30-63.

Evren, O. and E. A. Ok, On the multi-utility representation of preference relations, *J. Math. Econ.* 47 (2011), 554-563.

Evren, O., H. Nishimura, and E. A. Ok, Top Cycles and Revealed preference structures, *mimeo*,

https://hirokinishimura.com/wp-content/uploads/2021/05/RevPS.pdf.

Fishburn, P., Intransitive indifference in preference theory: A survey, *Operations Research* 18 (1970), 207-228.

Fishburn, P., Nontransitive preferences in decision theory, *J. Risk Uncertainty* 4 (1991), 113-134.

Frick, M., Monotone threshold representations, *Theoret. Econ.* 11 (2016), 757-772.

Galaabaatar, T., and E. Karni, Subjective expected utility with incomplete preferences, *Econometrica* 81 (2013), 255-284.

García-Sanz, M. and J. Alcantud, Sequential rationalization of multivalued choice, *Math. Soc. Sci.* 74 (2015), 29-33.

Giarlotta, A., New trends in preference, utility, and choice: from a mono-approach to a multi-approach, In: Doumpos M., Figueira J., Greco S., Zopounidis C. (eds) New Perspectives in Multiple Criteria Decision Making, Springer, (2019), 3-80.

Giarlotta, A., and S. Greco, Necessary and possible preference structures, *J. Math. Econ.* 49 (2013), 163-172.

Giarlotta, A., and S. Watson, A bi-preference interplay between transitivity and completeness: Reformulating and extending Schmeidler's theorem, *J. Math. Psych.* 96 (2020), 1-15.

Gilboa, I., and D. Schmeidler, Maxmin expected utility with non-unique prior, *J. Math. Econ.* 18 (1988), 141-153.

Gilboa, I., D. Schmeidler, F. Maccheroni, and M. Marinacci, Objective and subjective rationality in a multi-prior model, *Econometrica* 78 (2010), 755-770.

Hara, K., E. A. Ok, and G. Riella, Coalitional expected multi-utility theory, *Econometrica* 87 (2019), 933-980.

Heller, Y., Justifiable choice, *Games Econ. Behav.* 76 (2012), 375-390.

Kahneman, D. and A. Tversky, Prospect theory: An analysis of decision under risk, *Econometrica* 47 (1979), 263–292.

Kalai, E., and D. Schmeidler, An admissible set occuring in various bargaining situations, *J. Econ. Theory* 146 (1977), 402-411.

Laslier, J.-F., *Tournament Solutions and Majority Voting*, Studies in Economic Theory 7, Springer-Verlag, Heidelberg, 1997.

Lehrer, E., and R. Teper, Justifiable preferences, *J. Econ. Theory* 146 (2011), 762-774.

Lleras, J., Y. Masatlioglu, D. Nakajima, and E. Ozbay, When more is less: Limited consideration, *J. Econ. Theory* 170 (2017), 70-85.

Lombardi, M., Reason-based choice correspondences, *Math. Soc. Sci.* 57 (2009), 58-66.

Loomes, G., and Sugden, R., Regret theory: An alternative theory of rational choice under uncertainty, *Econ. J.* 92 (1982), 805-824.

Luce, D., Semiorders and a theory of utility discrimination, *Econometrica*, 24 (1956), 178-191.

Mandler, M., Incomplete preferences and rational intransitivity of choice, *Games Econ. Behav.* 50 (2005), 255-277.

Manzini, P., and M. Mariotti, Sequentially rationalizable choice, *Amer. Econ. Rev.* 97 (2007), 1824–

1839.

Masatlioglu, Y., D. Nakajima, and E. Ozbay, Revealed attention, *Amer. Econ. Rev.* 102 (2012), 2183-2205.

Nishimura, H. and E. A. Ok, Utility representation of an incomplete and nontransitive preference relation, *J. Econ. Theory* 166 (2016), 164-185.

Ok, E. A., Utility representation of an incomplete preference relation, *J. Econ Theory* 104 (2002), 429-449.

Ok, E. A., P. Ortoleva, and G. Riella, Incomplete preferences under uncertainty: Indecisiveness in beliefs versus tastes, *Econometrica* 80 (2012), 1791-1808.

Ok, E. A., P. Ortoleva, and G. Riella, Revealed (P)Reference Theory, *Amer. Econ. Rev.* 105 (2015), 299-321.

Riberio, M., and G. Riella, Regular preorders and behavioral indifference, *Theory Decis.* 82 (2017), 1-12.

Rubinstein, A., Similarity and decision-making under risk (is there a utility theory resolution to the Allais paradox?), *J. Econ. Theory* 46 (1988), 145-153.

Salant, Y., and A. Rubinstein, (A, f): Choice with frames, *Rev. Econ. Stud.* 75 (2008), 1287-1296.

Schwartz, T., Rationality and the myth of the maximum, *Noûs* 6 (1972), 97-117.

Schwartz, T., *The Logic of Collective Action*, Columbia University Press, New York, 1986.

Suzumura, K., Remarks on the theory of collective choice, *Economica*, 43 (1976), 381-390.

Tserenjigmid, G., Theory of decisions by intra-dimensional comparisons, *J. Econ. Theory* 159 (2015), 326-338.

Wallace, A., A fixed point theorem, *Bull. Amer. Math. Soc.* 51 (1945), 413-416.