Rational Decision Making with Preference Structures: A Behavioral Approach*

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Abstract

We model the preferences of a decision-maker by means of two binary relations. The first of these is transitive and captures comparisons that the agent finds easy or obvious. The second one describes the choices of the agent from binary choice problems, and it is assumed to be complete. Imposing two consistency conditions on these relations yields what we call a *preference structure*. The primary goal of the paper is to study the choice behavior that arises from preference structures. We show that the choice theory developed here, while much more general, embodies existence and uniqueness properties that parallel those of the classical choice theory. It also retains considerable predictive power.

JEL Classification: D11, D81.

Keywords: Preference structures, Incomplete and nontransitive preferences, behavioral economics, boundedly rational choice, top-cycles.

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1 Introduction

The preferences of an individual on a given set X of choice prospects is commonly modeled by means of a binary relation on X. In the classical approach this binary relation is assumed to be complete and transitive, and the alternatives that the agent finds choosable from any given feasible menu are those that maximize it.

This model is elegant and has considerable predictive power. Yet its explanatory scope is known to be limited. Experimental evidence documents violations of transitivity in observed choice behavior, which has led to the development of models of nontransitive preferences.[1] In turn, models of indecisiveness require relaxing the completeness hypothesis. Similar difficulties arise when the decision maker is a group (such as a board of directors). After all, the two most standard binary relations that are relevant in this case are the Pareto ordering (which is incomplete) and the majority voting rule (which is nontransitive).

A key limitation of the standard approach is that it treats all pairwise choice problems in the same way. In many real-life situations, however, some comparisons may be "easy," even "trivial," while others may be "difficult" enough to force one make unstable choices. Most of us would choose the sure lottery that pays \$10 over the one that pays \$5 "easily," while we may find comparing two complicated lotteries "difficult." Or, when a committee of experts unanimously advise that x is better than y, we are likely to regard deciding between x and y an "easy" problem, but if some favor x and others y, the problem may well become "hard." Similarly, comparing two social policies would be easy for a social planner when there is unanimous agreement about them in the society, but the choice problem may become "controversial" if some people back one policy, and others desire the alternate.

The upshot is that the choices across (subjectively) "hard" choice problems may be context dependent, thereby leading to a nontransitive choice pattern. (This viewpoint is not novel; see, among others, Mandler (2005) for a formal treatment, and Costa-Gomes et al. (2020) for empirical support.) By contrast, choices across "easy" pairwise problems (whichever these may be for the agent) are likely to abide by transitivity. But unless they are all "easy" for the agent, the pairwise problems yield only an incomplete ordering.

[1]See Day and Loomes (2010) for an overview of the evidence. Among the popular nontransitive preference models are regret theory (Loomes and Sugden (1982)), nontransitive indifference and similarity (Luce (1956), Fishburn (1970), Beja and Gilboa (1992), and Rubinstein (1988)), and framing effects (Kahneman and Tversky (1979) and Salant and Rubinstein (2008)).

A single binary relation cannot capture the distinction between "easy" and "hard" pairwise problems. In this paper we thus model preferences by means of two binary relations on X. The first of these, denoted as $\succsim$, represents comparisons that the decision maker regards clear-cut. We assume $\succsim$ is reflexive and transitive, but not necessarily complete. The second relation, denoted as $\mathbf{R}$, describes the observed choices of the agent in the case of all pairwise choice situations. This relation is, per force, complete, but it need not be transitive.

As $\succsim$ and $\mathbf{R}$ are meant to describe the preferences of a "rational" person, they must be consistent with each other. We thus assume both the weak and strict parts of $\mathbf{R}$ extend those of $\succsim$, respectively. In fact, we ask for a tighter coherence between $\succsim$ and $\mathbf{R}$ to prevent revealed choices to drift too far away from the agent's core preferences. Suppose $x \mathbf{R} y$ and $y \succsim z$ for some alternatives x, y and z. Thus, the agent declares x superior to y (although they may not be completely confident in this judgement) while they are sure that y is better than z. It then seems reasonable that they would prefer x over z, albeit, they may be insecure about this decision (that is, $x \mathbf{R} z$, but not necessarily $x \succsim z$). As the analogous reasoning applies also to the case where $x \succsim y$ and $y \mathbf{R} z$, it is natural to require $\mathbf{R}$ be *transitive with respect to* $\succsim$, which means $x \mathbf{R} z$ holds whenever $x \mathbf{R} y \succsim z$ or $x \succsim y \mathbf{R} z$.

Put precisely, a *preference structure* on X is a pair of binary relations $(\succsim, \mathbf{R})$ on X such that (i) $\succsim$ is a preorder on X, and (ii) $\mathbf{R}$ is a complete extension of $\succsim$ that is transitive with respect to $\succsim$. The goal of the present paper is to develop an operational theory of choice on the basis of such structures.[2]

Section 3 is devoted to a preliminary exposition of preference structures. We provide there several examples that highlight the scope of this model. These include models of (in)complete preferences, preferences with imperfect ability of discrimination, and preferences completed by the recommendations of a consultant. We then prove that for any preference structure $(\succsim, \mathbf{R})$, there is a set of preorders whose intersection is $\succsim$ and whose union is $\mathbf{R}$. Thus, the first component of any preference structure is a unanimity ordering, while its second component is a rationalizable preference in the sense of Cherepanov et al. (2013).

As we have noted above, the main concern of the present paper is developing a model of choice for a decision maker whose preferences are represented by a preference structure $(\succsim, \mathbf{R})$. This model is introduced in Section 4. For any feasible menu S,

[2] We emphasize that describing preferences through two binary relations is not new. We discuss the related literature at the end of Section 3.1. However, to the best of our knowledge, the present paper is the first to develop a systematic theory of choice based on such structures.

the interpretation of $\succsim$ maintains that the agent would not choose an alternative x from S if there is another alternative in S that strictly dominates x in terms of the "sure" ordering $\succsim$. Thus, the set $C(S)$ of all possible choices must be contained in $\mathbf{MAX}(S, \succsim)$, the set of all maximal elements in S with respect to $\succsim$. We then posit that the "choosable" alternatives in S should "maximize" $\mathbf{R}$ on $\mathbf{MAX}(S, \succsim)$. Of course, as $\mathbf{R}$ need not be transitive, there is no *a priori* reason for the existence of such maxima, even when S contains only three alternatives. This is a problem familiar from social choice theory, and it is often addressed by using an alternative notion of optima. Due to its intuitive appeal (Section 4.1), we adopt the top-cycle solution here, and set

$$C(S) := \text{top-cycle in } \mathbf{MAX}(S, \succsim) \text{ with respect to } \mathbf{R}.$$

This model generalizes the standard rational choice model, but its descriptive scope is larger. In particular, we show that it captures many interesting choice frameworks, among which are the models of rational choice with incomplete preferences, and various types of satisficing models. It is nevertheless a model of "rational choice," and as we demonstrate in Section 4.6, it packs a considerable amount of predictive power.

One of the main reasons why nontransitive preferences are seldom used in economics is that they may lead to empty-valued choice correspondences, leaving one with no predictions in some situations. Fortunately, on this score, our model matches those of the rational choice model exactly. Our first main theorem (Theorem 4.3) shows that under the usual compactness and continuity hypotheses, any choice correspondence that is rationalized by a preference structure is sure to be nonempty-valued.

In Section 5, we turn to the matter of observability of preferences from choices. Suppose C is the observed choice correspondence of a person, and assume that it is rationalized by some (unknown) preference structure $(\succsim, \mathbf{R})$. The question is to what extent one can elicit the agent's preference structure from C. This is of import, because the core preference relation of the agent is unobservable, but it is this part of the preference structure of the agent that matters most for welfare analysis (in the sense of, say, Bernheim and Rangel (2007, 2009)).

Section 5 is devoted to this issue. We first observe that the revealed preference part $\mathbf{R}$ of agent's preference structure is uniquely identified from her C. Next, we show that while there may well be a multitude of (incomplete) preference relations that, when coupled with $\mathbf{R}$, would rationalize C, the set $\mathbb{P}(C)$ of all such (core) preferences possesses an unexpected structure. Our second main result (Theorem 5.2) shows that any collection of such preferences can be combined into a single preference relation which, when coupled with $\mathbf{R}$, rationalizes C. Consequently, there is a *most* *decisive* preference relation $\succsim_C$ in $\mathbb{P}(C)$. Choice theoretically, $(\succsim, \mathbf{R})$ and $(\succsim_C, \mathbf{R})$ are "equivalent." Further, $\succsim_C$ is the preorder that allows us to make unambiguous welfare comparisons most frequently, and it is perfectly observable.

Our final main result gives an explicit characterization for $\succsim_C$ (Theorem 5.3). It turns out that the strict part of this relation corresponds precisely to the Bernheim-Rangel criterion (that is, $x \succ_C y$ iff y is never chosen in a feasible set that contains x). Its symmetric part renders two alternatives indifferent iff these alternatives are behaviorally equivalent in the sense of Eliaz and Ok (2006). In sum, from any choice correspondence that is rationalizable by a preference structure, we can recover the preference structure (i) which rationalizes that correspondence; and (ii) whose core part exhibits the least amount of indecisiveness compatible with that choice correspondence. Demonstration of this concludes our exposition.

In passing, we emphasize that our approach in this paper is that of behavioral economics. We outline a theory of choice, demonstrate how this theory extends the scope of the classical one, look at its basic implications, and work out its existence and uniqueness properties. An alternative, complementary approach would be that of axiomatic decision theory. This would ask for the determination of the behavioral content of our choice theory in terms of a complete axiomatic system. While this approach is essential, we do not adopt it here. This is only for brevity. A complete characterization of the present choice model in the tradition of revealed preference theory is provided in a separate, companion paper by Evren, Nishimura and Ok (2026).

2 Nomenclature

Binary Relations. By a *binary relation* on a nonempty set X, we mean any nonempty subset $\mathbf{R}$ of $X \times X$, but write $x \mathbf{R} y$ instead of $(x, y) \in \mathbf{R}$. For any nonempty $S \subseteq X$, by $x \mathbf{R} S$, we mean $x \mathbf{R} y$ for every $y \in S$. Moreover, for any binary relations $\mathbf{R}$ and $\mathbf{S}$ on X, we simply write $x \mathbf{R} y \mathbf{S} z$ to mean $x \mathbf{R} y$ and $y \mathbf{S} z$, and so on. For any nonempty $S \subseteq X$, the *restriction* of $\mathbf{R}$ to S is the binary relation on S given by $\mathbf{R}|_S := \mathbf{R} \cap (S \times S)$.

For any $x \in X$, the $\mathbf{R}$-*upper set* of x is defined as $x^{\uparrow,\mathbf{R}} := \{y \in X : y \mathbf{R} x\}$, and the $\mathbf{R}$-*lower set* of x is $x^{\downarrow,\mathbf{R}} := \{y \in X : x \mathbf{R} y\}$. When either $x \mathbf{R} y$ or $y \mathbf{R} x$, we say that x and y are $\mathbf{R}$-*comparable*, and define

$$\text{Inc}(\mathbf{R}) := \{(x, y) \in X \times X : x \text{ and } y \text{ are not } \mathbf{R}\text{-comparable}\}.$$

If $\text{Inc}(\mathbf{R}) = \varnothing$, we say that $\mathbf{R}$ is *complete* (or *total*).

The *asymmetric* (or *strict*) *part* of a binary relation $\mathbf{R}$ on X is the binary relation $\mathbf{R}^>$ on X with $x\ \mathbf{R}^>\ y$ iff $x\ \mathbf{R}\ y$ and not $y\ \mathbf{R}\ x$, and the *symmetric part* of $\mathbf{R}$ is defined as $\mathbf{R}^= := \mathbf{R} \backslash \mathbf{R}^>$. The *composition* of two binary relations $\mathbf{R}$ and $\mathbf{S}$ on X is defined as $\mathbf{R} \circ \mathbf{S} := \{(x, y) \in X \times X : x\ \mathbf{R}\ z\ \mathbf{S}\ y \text{ for some } z \in X\}$. We say that $\mathbf{S}$ is a *subrelation* of $\mathbf{R}$, and that $\mathbf{R}$ is a *superrelation* of $\mathbf{S}$, if $\mathbf{S} \subseteq \mathbf{R}$.

We denote the diagonal of $X \times X$ by $\triangle_X$, that is, $\triangle_X := \{(x, x) : x \in X\}$. A binary relation $\mathbf{R}$ on X is said to be *reflexive* if $\triangle_X \subseteq \mathbf{R}$, *antisymmetric* if $\mathbf{R}^= \subseteq \triangle_X$, *transitive* if $\mathbf{R} \circ \mathbf{R} \subseteq \mathbf{R}$, and *quasitransitive* if $\mathbf{R}^>$ is transitive. If $\mathbf{R}$ is reflexive and transitive, we refer to it as a *preorder* on X. (Throughout the paper, generic preorders are denoted as $\succsim$ or $\trianglerighteq$, whose asymmetric parts are denoted as $\succ$ and $\triangleright$, respectively.) Finally, an antisymmetric preorder on X is said to be a *partial order* on X. If X is endowed with a prespecified partial order, we may refer to it as a *poset*.

The *transitive closure* of a binary relation $\mathbf{R}$ on X is the smallest transitive superrelation of $\mathbf{R}$; we denote it by $\text{tran}(\mathbf{R})$. We have $x\ \text{tran}(\mathbf{R})\ y$ iff there exist a $k \in \mathbb{Z}_+$ and $x_0, ..., x_k \in X$ such that $x = x_0\ \mathbf{R}\ x_1\ \mathbf{R} \cdots \mathbf{R}\ x_k = y$. Obviously, $\text{tran}(\mathbf{R})$ is a preorder on X, provided that $\mathbf{R}$ is reflexive.

Extension of Binary Relations. Let $\mathbf{R}$ be a binary relation on X. If $\mathbf{S}$ and $\mathbf{S}^>$ are subrelations of $\mathbf{R}$ and $\mathbf{R}^>$, respectively, we say that $\mathbf{R}$ is an *extension* of $\mathbf{S}$ (or that $\mathbf{R}$ *extends* $\mathbf{S}$). If $\mathbf{R}$ extends $\mathbf{S}$ and it is total, we refer to it as a *completion* of $\mathbf{S}$.

Transitivity with Respect to another Binary Relation. Our main focus in this paper is on reflexive, but not necessarily transitive, binary relations. A useful concept in the analysis of such binary relations is the notion of *transitivity with respect to a binary relation*. Put precisely, given any two binary relations $\mathbf{R}$ and $\mathbf{S}$ on X, we say that $\mathbf{R}$ is $\mathbf{S}$-*transitive* if $\mathbf{R} \circ \mathbf{S} \subseteq \mathbf{R}$ and $\mathbf{S} \circ \mathbf{R} \subseteq \mathbf{R}$.

3 Preference Structures

3.1 Introduction

Let X be a nonempty set which we take as the collection of all mutually exclusive choice prospects for a decision-making unit, or an agent, who may itself be a collection of individuals (such as a board of directors, congress, or a family). This agent is entirely confident in the preferential ranking of *some* of the alternatives in X. We model this by means of a preorder $\succsim$ on X. So, when $x \succsim y$, we understand that the agent is "sure" that x is better than y for her. Of course, $\succsim$ is unobservable (because we do not know when an agent is "sure" about her preferential rankings), but our interpretation justifies its transitivity. However, and this is where we begin to deviate from the standard theory of rational decision-making, there is no need for $\succsim$ to be complete.[3]

Suppose the agent is unable to rank two alternatives x and y with respect to $\succsim$. When confronted with the problem of choosing between x and y, one will nevertheless observe they make a decision.[4] So, in the case of this choice problem, if they choose x over y, we say that "x is revealed preferred to y," and if we have somehow witnessed that they choose x over y at some point, and y over x in some other, we say that "x is revealed indifferent to y." As such, we model *all* pairwise rankings of the individual, "easy" ones as well as the "hard" ones, by means of a binary relation $\mathbf{R}$ on X. The interpretation of $\mathbf{R}$ mandates it be complete. However, it is only natural that "hard choices" may not act transitively. When the problem is tough, one may use context dependent method of settling it, and this may well give rise to nontransitive choice patterns. Moreover, if our economic agent consists of a set of individuals, then even the most standard methods of aggregating constituent preferences (such as majority voting) may result in the revelation of nontransitive rankings.

We thus model the "preferences" of an economic agent by means of an ordered pair $(\succsim, \mathbf{R})$ of binary relations on X such that $\succsim$ is a preorder and $\mathbf{R}$ is complete. Moreover, for consistency, we assume $x \succsim y$ implies $x \mathbf{R} y$; this simply means that if x is "surely" at least as desirable as y for the agent, we would observe her choose x over y. It is actually meaningful to ask $\mathbf{R}$ act in coherence with $\succsim$ in a way that goes beyond this property. Suppose our agent declares $x \mathbf{R} y$ and $y \succsim z$. That is, they like x better than y, even though they may well be somewhat insecure about this decision, while they prefer y over z in complete confidence. It is then sensible that the "obvious" superiority of y over z would give a reason to the agent to prefer x over z, but, of course, they may not be fully confident about this (that is, $x \mathbf{R} z$, but not necessarily $x \succsim z$). Consequently, and since the same reasoning applies when $x \succsim y$

[3]For instance, the agent may be employing a committee of experts, or they may be a social planner on behalf of a collection of individuals (each with her own preference relation). In either of these cases, $\succsim$ may correspond to the rankings of prospects according to the unanimity rule. When this rule applies, the comparisons are "easy," but there may be cases in which unanimity fails.

[4]In principle, the agent may "choose" not to make a choice, but this necessitates that at least some pairwise choice problems (those that do not include the option of not choosing) to be designated as unobservable situations. As formalized later, we abstract away from such contingencies by tacitly allowing all pairwise choice situations within our framework.

$\mathbf{R}\ z$ as well, it makes good sense to require $\succsim$ and $\mathbf{R}$ to satisfy the following:

$$x\ \mathbf{R}\ y \succsim z \quad \text{or} \quad x \succsim y\ \mathbf{R}\ z \quad \text{implies} \quad x\ \mathbf{R}\ z$$

for all $x, y, z \in X$. Put succinctly, we posit $\mathbf{R}$ be $\succsim$*-transitive.* This property is not only reasonable, but it also brings some discipline to the model, and allows us learn quite a bit about $\succsim$ (which is unobservable) from $\mathbf{R}$ (which is observable).

These considerations prompt the following:

Definition. An ordered pair $(\succsim, \mathbf{R})$ is a *weak preference structure* on a nonempty set X if $\succsim$ is a preorder on X and $\mathbf{R}$ is a $\succsim$-transitive and complete superrelation of $\succsim$ on X. In this context, we refer to $\succsim$ as the *core preference relation* of the structure, and to $\mathbf{R}$ as its *revealed preference relation.*

There are two special cases of the notion of a weak preference structure $(\succsim, \mathbf{R})$ that are of immediate interest. The first one of these strengthens the connection between the core and revealed preferences by requiring $\mathbf{R}$ be an *extension* of $\succsim$. This amounts to requiring $x \succ y$ imply $x\ \mathbf{R}^{>}\ y$ (in addition to $\succsim\ \subseteq \mathbf{R}$), that is, if x is "surely" strictly better than y for the agent, then they would never choose y over x.[5] The second one, instead, asks for $\mathbf{R}$ itself be transitive. This model embodies a lot of rationality within, but as we shall see, it still entails a choice theory distinct from the classical rational choice theory. Furthermore, instances of this version of weak preference structures have already been considered in the literature.

Definition. A weak preference structure $(\succsim, \mathbf{R})$ on a nonempty set X is said to be a *preference structure* on X, provided that $\mathbf{R}$ is a completion of $\succsim$.[6] In turn, a (weak) preference structure is said to be a *transitive* (*weak*) *preference structure* on X, if $\mathbf{R}$ is transitive.

Relation to the Literature. Modeling individual preferences by means of two binary relations, one incomplete and the other complete, is not new in decision theory. Especially in the literature on decision making under uncertainty, this method is employed

[5] This requirement is only natural. For, suppose $\succsim\ \subseteq \mathbf{R}$ holds, but $\succ\ \subseteq \mathbf{R}^{>}$ fails. Then, even though it is "obvious" to the agent that x is strictly better than y (that is $x \succ y$), we may have $x\ \mathbf{R}^{=}\ y$ which means that the agent could choose y over x at some point. For instance, this would leave room for a person to surely like receiving \$1000 over \$0, but still to choose \$0 over \$1000 every now and again!

[6] In what follows, when we wish to emphasize that a preference structure is not weak, we will refer to it as a *proper* preference structure.

by a number of studies. For example, in the Anscombe-Aumann framework, Gilboa et al. (2010) have used two binary relations, the first being a preorder (à la Bewley (1986)) and the second a complete preorder (à la Gilboa and Schmeidler (1989)). In the jargon introduced above, that model is a transitive weak preference structure.

In a general framework, both Mandler (2005) and Danan (2008) suggest distinguishing between one's core preferences – Mandler refers to these as *psychological preferences*, and Danan as *cognitive preferences* – from their revealed preferences. Mandler's model, whose outcome space is an open subset of $\mathbb{R}^n_+$, is, essentially, a special type of preference structure – see Example 3.3 below – but Mandler's emphasis is on sequential, nontransitive choice that is nevertheless consistent with core preferences. By contrast, Danan's model is of the form $(\succsim, \mathbf{R})$, where $\succsim$ and $\mathbf{R}$ are binary relations on (a topological space) X such that both $\succsim$ and $\mathbf{R}$ are complete, and $\succ \subseteq \mathbf{R}^>$. Danan uses this model to suggest a method of understanding when an individual who has been observed to choose x over y is, in fact, indifferent between x and y. Notably, his model is a preference structure iff $\succsim = \mathbf{R}$, that is, in the intersection of our model and Danan's lies only the classical model of preferences.

The two models of preferences that are closest to the one we consider here are those of Giarlotta and Greco (2013) and Giarlotta and Watson (2020). Both of these papers work with a weak preference structure $(\succsim, \mathbf{R})$ on X. Giarlotta and Greco (2013) impose the following additional requirement on this model: For any two alternatives x and y, either $x \succsim y$ or $y \mathbf{R} x$. This model, which is called the *necessary and possible preference* on X, declares any two alternatives that are "hard" to compare as revealed indifferent. As such, it appears rather restrictive to serve as a general model of individual preferences (but it has been useful in multi-criteria decision analysis; see Giarlotta (2019) for a survey.) Giarlotta and Watson (2020) consider, instead, imposing the (mutually exclusive) requirements $\succ \subseteq \mathbf{R}^>$ or $\mathbf{R}^> \subseteq \succ$ on $(\succsim, \mathbf{R})$. In their jargon, the first of these leads to *complete monotonic bi-preferences* and the second to *complete comonotonic bi-preferences*. While the former model is identical to what we define here as preference structures, Giarlotta and Watson (2020) instead focus on exploring the structure of the latter model.

As we have noted in Section 1, our focus in the present paper is not a preference structure per se. Instead, our goal is to explore how preference structures could be used to model the "choices" of an agent in a way that generalizes the classical rational choice theory. We are not aware of any work that studies this issue.

3.2 Examples

We next give concrete examples of preference structures on a nonempty set X.

Example 3.1. Let $\succsim$ be a complete preorder on X. Then, $(\succsim, \succsim)$ is a transitive preference structure on X. (Every complete preference relation may thus be thought of as a preference structure.)

Example 3.2. Let $\mathbf{R}$ be a total binary relation on X. Then, $(\triangle_X, \mathbf{R})$ is a preference structure on X. (Every total binary relation may thus be thought of as a preference structure.)

Example 3.3. Let $\succsim$ be a preorder on X. Then, $(\succsim, \succsim \sqcup \mathrm{Inc}(\succsim))$ is a preference structure on X. (This is, essentially, the model Mandler (2005) has considered in the context of consumer choice.)

Example 3.4. (*Aggregation by Social Welfare Criteria*) Take any $n \in \mathbb{N}$, and $u_i : X \to \mathbb{R}$, $i = 1, ..., n$. On X, define a preorder $\succsim$ by $x \succsim y$ iff $u_i(x) \geq u_i(y)$ for each i, and the binary relations $\mathbf{R}^1$, $\mathbf{R}^2$, and $\mathbf{R}^3$ by

$$x\,\mathbf{R}^1\,y \quad \text{iff} \quad \sum_{i=1}^{n} u_i(x) \geq \sum_{i=1}^{n} u_i(y),$$

$$x\,\mathbf{R}^2\,y \quad \text{iff} \quad \min_{i=1,\ldots,n} u_i(x) \geq \min_{i=1,\ldots,n} u_i(y),$$

and

$$x\,\mathbf{R}^3\,y \quad \text{iff} \quad \max_{i=1,\ldots,n} (u_i(x) - u_i(y)) \geq 0,$$

respectively. We may think of $\succsim$ here as a *Pareto ordering*, while $\mathbf{R}^1$ and $\mathbf{R}^2$ correspond to the *utilitarian* and *Rawlsian* social welfare criteria, respectively. In turn, $\mathbf{R}^3$, which often is nontransitive, is best viewed as a *justifiable preference* (to borrow the jargon used by Lehrer and Teper (2011)). Each $(\succsim, \mathbf{R}^i)$ is a weak preference structure on X. In fact, $(\succsim, \mathbf{R}^1)$ is a transitive preference structure, while $(\succsim, \mathbf{R}^2)$ is a transitive weak preference structure (but it need not be a preference structure). By contrast, $(\succsim, \mathbf{R}^3)$ need not be a preference structure, nor need it be transitive.

Example 3.5. (*Aggregation by Majority Voting*) Let $\mathcal{P}$ be a nonempty finite family of total preorders on X. Then, $\bigcap \mathcal{P}$ is the Pareto ordering induced by this family. In turn, we define the (*majority voting*) binary relation $\mathcal{P}_{\mathrm{maj}}$ on X as

$$x\ \mathcal{P}_{\mathrm{maj}}\ y \quad \text{iff} \quad |\{\succsim \in \mathcal{P} : x \succ y\}| \geq |\{\succsim \in \mathcal{P} : y \succ x\}|$$

for every $x, y \in X$. Then, $(\bigcap \mathcal{P}, \mathcal{P}_{\text{maj}})$ is a preference structure on X.

Example 3.6. (*Cautious Expected Utility Theory*) Let $\mathcal{U}$ be a family of continuous and strictly increasing real maps on $[0,1]$, and define the binary relation $\succsim$ on $\Delta[0,1]$, the set of all Borel probability measures on $[0,1]$, with

$$p \succsim q \quad \text{iff} \quad \int_{[0,1]} u \, \mathrm{d}p \geq \int_{[0,1]} u \, \mathrm{d}q \text{ for all } u \in \mathcal{U}.$$

In the terminology of Dubra, Maccheroni and Ok (2004), $\succsim$ is a preorder on $\Delta[0,1]$ that admits an *expected multi-utility representation.* Next, consider the complete preorder $\mathbf{R}$ on $\Delta[0,1]$ with

$$p \, \mathbf{R} \, q \quad \text{iff} \quad \inf_{u \in \mathcal{V}} \left(\int_{[0,1]} u \, \mathrm{d}p \right) \geq \inf_{u \in \mathcal{V}} \left(\int_{[0,1]} u \, \mathrm{d}q \right).$$

In the terminology of Cerreia-Vioglio, Dillenberger and Ortoleva (2015), $\mathbf{R}$ admits a *cautious expected utility representation.* $(\succsim, \mathbf{R})$ is a transitive weak preference structure on X. It need not be a preference structure, however.

Example 3.7. (*Preferences with Imperfect Discrimination*) Let $\mathbf{R}$ be a complete and quasitransitive binary relation on X. Then, $(\triangle_X \sqcup \mathbf{R}^>, \mathbf{R})$ is a preference structure on X. This model captures the utility model of imperfect discrimination which goes back to Armstrong (1939) and Luce (1956), and is studied more recently by Beja and Gilboa (1992), among others. To wit, let $u : X \to \mathbb{R}$ be any function and take any $\varepsilon \geq 0$. Define the binary relation $\mathbf{R}$ on X as $x \, \mathbf{R} \, y$ iff $u(x) \geq u(y) - \varepsilon$. This is a complete and quasitransitive binary relation on X with $x \, \mathbf{R}^> \, y$ iff $u(x) > u(y) + \varepsilon$ and $x \, \mathbf{R}^= \, y$ iff $|u(x) - u(y)| \leq \varepsilon$. Then, $(\succsim, \mathbf{R})$ is a preference structure on X where $\succsim$ is the semiorder with $x \succsim y$ iff either $x = y$ or $u(x) > u(y) + \varepsilon$. The interpretation is that the pairwise ranking of any two alternatives is an "easy" one if the utilities of these alternatives are sufficiently distinct, and "hard" otherwise.

Example 3.8. (*Intra-Dimensional Comparison Heuristics*) Let $n \in \mathbb{N}$, and consider an environment in which every commodity is modeled through n attributes. We thus interpret $x := (x_1, ..., x_n) \in \mathbb{R}^n$ as a good that possesses x_i units of the attribute i. For each i, let us pick any skew-symmetric function $f_i : \mathbb{R}^2 \to (-1, 1)$ that is strictly increasing in the first component, and any strictly increasing and odd map $W : (-1,1)^n \to \mathbb{R}$. We define the binary relation $\mathbf{R}$ on $\mathbb{R}^n$ as $x \, \mathbf{R} \, y$ iff $W(f_1(x_1, y_1), ..., f_n(x_n, y_n)) \geq 0$. Here the vector $(f_1(x_1, y_1), ..., f_n(x_n, y_n))$ corresponds to the comparison of goods x and y attribute by attribute; we can interpret f_i

as measuring either the (dis)similarity of x_i and y_i or the salience of the ith attribute relative to the other attributes. Tserenjigmid (2015) calls $\mathbf{R}$ an *intra-dimensional comparison* (IDC) *relation*, and show that regret preferences of Loomes and Sugden (1982), the additive difference model of Tversky (1969), and a version of the salience theory of Bordalo, Gennaioli and Schleifer (2012), are of this form. Note that $(\succsim, \mathbf{R})$ is a preference structure on X, where $x \succsim y$ iff $f_i(x_i, y_i) \geq 0$ for each $i = 1, ..., n$.

3.3 Weak vs. Proper Preference Structures

The only reason why a weak preference structure $(\succsim, \mathbf{R})$ on X may not be a preference structure is that there may be alternatives x and y in X such that the agent inherently prefers x over y strictly (that is, $x \succ y$) and yet the revealed preference $\mathbf{R}$ views x and y equally desirable (that is, $x\ \mathbf{R}^=\ y$). If, therefore, we refine $\mathbf{R}$ so as to drop (y, x) from it (for any such x and y), we obtain a preference structure. This provides a natural way of obtaining a preference structure from a weak preference structure. Formally, we define the binary relation $\mathbf{R}_{\succsim}$ on X with $x\, \mathbf{R}_{\succsim}\, y$ iff

$$\text{either } x \succsim y \text{ or } [x \text{ and } y \text{ are not } \succsim\text{-comparable and } x\ \mathbf{R}\ y]. \tag{1}$$

Coupling $\succsim$ with $\mathbf{R}_{\succsim}$ yields a preference structure:

Proposition 3.1. *Let $(\succsim, \mathbf{R})$ be a weak preference structure on a nonempty set X. Then, $(\succsim, \mathbf{R}_{\succsim})$ is a preference structure on X.*

Proper Representation. We refer to $(\succsim, \mathbf{R}_{\succsim})$ as the *proper representation* of the weak preference structure $(\succsim, \mathbf{R})$. (Of course, if $(\succsim, \mathbf{R})$ is proper to begin with, we have $(\succsim, \mathbf{R}_{\succsim}) = (\succsim, \mathbf{R})$.) In Section 4, we will show that $(\succsim, \mathbf{R})$ and its proper representation $(\succsim, \mathbf{R}_{\succsim})$ are equivalent in a formal, choice-theoretic sense.

A transitive weak preference structure need not be a preference structure (Examples 3.4 and 3.6). Interestingly, the proper representation of such a structure may lose its transitivity. However, again in terms of the choice theory that we will introduce in Section 4, this does not make a difference, because the revealed preference part of the proper representation is quasitransitive.

Corollary 3.2. *Let $(\succsim, \mathbf{R})$ be a transitive weak preference structure on a nonempty set X. Then, $(\succsim, \mathbf{R}_{\succsim})$ is a preference structure on X and $\mathbf{R}_{\succsim}$ is quasitransitive.*

Modeling Choice by Consultation. There is a nice interpretation of the image of a transitive weak preference structure under the proper representation. Think of an individual who, when faced with a "hard" choice problem, seeks the advise of a consultant. So, when two alternatives x and y are not comparable by $\succsim$ – this choice is "hard" for the agent – they ask another agent what to do. Imagine that the consultant is rational in the traditional sense, so their advice stems from a complete preorder $\mathbf{R}$ on X. Further, assume that this preorder is consistent with $\succsim$ in the sense that $\succsim \subseteq \mathbf{R}$. (Otherwise, it would be unrealistic to presume that the agent trust the recommendations of the advisor, as some of those would conflict with their core preferences.) As such, $(\succsim, \mathbf{R})$ is a transitive weak preference structure on X. But, in this interpretation, $\mathbf{R}$ does not correspond to the revealed preferences of the agent. After all, these preferences reflect those of the consultant only over the problems that our agent seeks help for. In other words, the revealed preferences of the agent coincide with $\succsim$ whenever $\succsim$ is able to render a ranking, and with $\mathbf{R}$ when this is not possible. The preference structure $(\succsim, \mathbf{R})$ corresponds to this interpretation. In this model, the revealed preferences of the agent need not be transitive, but they are quasitransitive.

3.4 Characterization of Preference Structures

The following result provides a representation that connects the two components of a preference structure by means of a single family of preorders.

Theorem 3.3. *Let $\succsim$ and $\mathbf{R}$ be binary relations on a nonempty set X. Then, $(\succsim, \mathbf{R})$ is a (weak) preference structure on X if, and only if, there is a nonempty collection $\mathcal{P}$ of preorders on X such that*

$$(\succsim, \mathbf{R}) = \left(\bigcap \mathcal{P}, \bigcup \mathcal{P} \right) \tag{2}$$

where $\bigcup \mathcal{P}$ is complete and each $\trianglerighteq \in \mathcal{P}$ extends (includes) $\succsim$.[7]

The "if' part of this result provides a general method of defining preference structures. In turn, its "only if" part provides a *multi-selves* interpretation. To wit, let $\mathcal{P}$ stand for a family of preorders on X as found in Theorem 3.3. We may think of each member of $\mathcal{P}$ as a (potentially incomplete) preference relation of a different "self" of the same individual. (For instance, the agent may not know which of these relations will be relevant at the time of consumption, so entertains them all before making their choice.) These "selves" of the agent are consistent with the core preference relation $\succsim$ as each of them extends $\succsim$. In addition, $\succsim$, being equal to $\bigcap \mathcal{P}$, corresponds to the *unanimity* of them. On the other hand, the revealed preference relation $\mathbf{R}$ of

[7] A similar result for necessary and possible preferences is obtained by Giarlotta and Greco (2013).

the agent equals $\bigcup \mathcal{P}$, so it ranks x over y iff at least one of their "selves" agrees to this ranking. In this sense, we may think of $\mathbf{R}$ as a *rationalizable* preference on X, borrowing (and slightly abusing) the terminology used by Cherepanov, Feddersen and Sandroni (2013). Importantly, these notions of unanimity and rationalizability are compatible, for they are based on the preferences of the same set of "selves."[8]

We conclude by noting that Theorem 3.3 modifies readily to give a characterization of transitive preference structures.

Corollary 3.4. *Let $\succsim$ and $\mathbf{R}$ be binary relations on X with $\mathbf{R}$ being complete. Then, $(\succsim, \mathbf{R})$ is a transitive (weak) preference structure on X if, and only if, there is a nonempty collection $\mathcal{P}$ of preorders on X such that (2) holds, $\mathbf{R} \in \mathcal{P}$, and each $\trianglerighteq \in \mathcal{P}$ extends (includes) $\succsim$.*

4 Choice by Preference Structures

We now turn to our main query: How do preference structures give rise to rational choices? This requires us agree on what it means for an alternative to "maximize" a complete binary relation on a feasible set, so we first address this issue.

4.1 Maximization of Complete Binary Relations

Let X be a nonempty set, $\mathbf{R}$ a binary relation on X, and S a nonempty subset of X. An element x of S is called $\mathbf{R}$-*maximal* in S if there is no $y \in S$ with $y \ \mathbf{R}^{>} \ x$, and $\mathbf{R}$-*maximum* in S if $x \ \mathbf{R} \ S$. We denote the set of all $\mathbf{R}$-maximal and $\mathbf{R}$-maximum elements in S by $\mathbf{MAX}(S, \mathbf{R})$ and $\max(S, \mathbf{R})$, respectively. We always have $\max(S, \mathbf{R}) \subseteq \mathbf{MAX}(S, \mathbf{R})$, but this inequality may hold strictly (unless $\mathbf{R}$ is complete).

When $\mathbf{R}$ is not transitive, $\mathbf{MAX}(S, \mathbf{R})$ may be empty even for a finite set S. For this reason, alternative notions of extrema are developed for binary relations. The best-known of these is the notion of *top-cycles*.

Top-Cycles. Let $\mathbf{R}$ be a complete binary relation on X. We say that a nonempty subset

[8] The reader may ask if one can guarantee the completeness of each member of $\mathcal{P}$ in the representation provided in Theorem 3.3. Unfortunately, this is a very restrictive requirement; we can do this only when $\mathbf{R}$ is obtained from $\succsim$ by rendering every $\succsim$-incomparable pair indifferent. That is, a preference structure $(\succsim, \mathbf{R})$ on X satisfies $\mathbf{R} = \succsim \sqcup \text{Inc}(\succsim)$ iff there is a nonempty collection $\mathcal{P}$ of complete preorders on X such that (i) $(\succsim, \mathbf{R}) = (\bigcap \mathcal{P}, \bigcup \mathcal{P})$, and (ii) each $\trianglerighteq \in \mathcal{P}$ extends $\succsim$. (We omit the proof, which is available upon request.)

A of S is a *highset in S with respect to* $\mathbf{R}$, or more simply, an $\mathbf{R}$-*highset in* S, if

$$x \ \mathbf{R}^{>} \ y \quad \text{for every } x \in A \text{ and } y \in S \backslash A.$$

The collection of all $\mathbf{R}$-highsets in S is nonempty, because it contains S. Moreover, this collection is linearly ordered by set inclusion.[9] Consequently, if it exists, there is a unique smallest $\mathbf{R}$-highset in S, namely, the intersection of all $\mathbf{R}$-highsets in S. We thus define the *top-cycle in S with respect to* $\mathbf{R}$ as

$$\bigcirc(S, \mathbf{R}) := \bigcap\{A : A \text{ is an } \mathbf{R}\text{-highset in } S\}.$$

This set is nonempty iff the smallest $\mathbf{R}$-highset in S exists. In particular, $\bigcirc(S, \mathbf{R}) \neq \varnothing$ whenever S is finite.[10]

By "maximization of $\mathbf{R}$ in S," we mean identifying $\bigcirc(S, \mathbf{R})$. Not only is this intuitive, but it is consistent with the standard case (for, $\bigcirc(S, \mathbf{R}) = \max(S, \mathbf{R})$ whenever $\mathbf{R}$ is transitive). The following shows that top-cycles indeed correspond to a well-defined optimization principle.

Proposition 4.1. *Let S be a nonempty subset of a set X, and $\mathbf{R}$ a complete binary relation on X. Then,*

$$\bigcirc(S, \mathbf{R}) = \max(S, \mathrm{tran}(\mathbf{R}|_S)).$$

A nonempty subset A of X is an $\mathbf{R}$-*cycle* if for any $x, y \in A$, there are finitely many $a_1, ..., a_k \in A$ with $x \ \mathbf{R} \ a_1 \ \mathbf{R} \cdots \mathbf{R} \ a_k \ \mathbf{R} \ y$. (If A is finite and $\mathbf{R}$ is complete, then A is an $\mathbf{R}$-cycle iff we can enumerate A as $\{x_1, ..., x_n\}$ so that $x_1 \ \mathbf{R} \ x_2 \ \mathbf{R} \cdots \mathbf{R} \ x_n \ \mathbf{R} \ x_1$.) The following observation, which generalizes a theorem of Schwartz (1972), characterizes $\bigcirc(S, \mathbf{R})$ as the unique $\mathbf{R}$-highset in S that is also an $\mathbf{R}$-cycle.

Corollary 4.2. *Let S and T be nonempty subsets of X, and $\mathbf{R}$ a complete binary relation on X with $\bigcirc(S, \mathbf{R}) \neq \varnothing$. Then, $T = \bigcirc(S, \mathbf{R})$ if, and only if, T is both an $\mathbf{R}$-highset in S and an $\mathbf{R}$-cycle.*

Proof. Put $T := \bigcirc(S, \mathbf{R})$. Then, T is an $\mathbf{R}$-highset in S. Further, by Proposition 4.1, if $x, y \in T$, then $x \, \mathrm{tran}(\mathbf{R}|_S)^{=} y$, so there exist $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in S$ with

[9] *Proof.* Suppose A and B are two $\mathbf{R}$-highsets in S with $A \subseteq B$ false. Then, pick any $a \in A \backslash B$, and notice that, for any $b \in B$, we have $b \ \mathbf{R}^{>} \ a$ because $a \in S \backslash B$ and B is an $\mathbf{R}$-highset in S. As A is itself an $\mathbf{R}$-highset in S, and $a \in A$, this implies $b \in A$ for each $b \in B$.

[10] Top-cycles are studied extensively in the case where $\mathbf{R}$ is a tournament (that is, an asymmetric total binary relation on a finite set). See, for instance, Laslier (1997).

$x \, \mathbf{R} \, a_1 \, \mathbf{R} \cdots \mathbf{R} \, a_k \, \mathbf{R} \, y$, that is, T is an $\mathbf{R}$-cycle. Conversely, suppose T is both an $\mathbf{R}$-highset in S and an $\mathbf{R}$-cycle. The first of these hypotheses implies $x \, \mathrm{tran}(\mathbf{R}|_S)^> \, y$ for any $x \in T$ and $y \in S \backslash T$, whereas the second implies $x \, \mathrm{tran}(\mathbf{R}|_S)^= \, y$ for all $x, y \in T$. Put together, we get $T = \max(S, \mathrm{tran}(\mathbf{R}|_S))$. By Proposition 4.1, we are done. ■

4.2 Rationalization by Preference Structures

Let X be a nonempty set, and $\mathfrak{X}$ any collection of nonempty subsets of X such that (i) $\mathfrak{X}$ contains all singletons, and (ii) $\mathfrak{X}$ is closed under taking finite unions. (In particular, $\mathfrak{X}$ contains all nonempty finite subsets of X). We call to any such ordered pair $(X, \mathfrak{X})$ a *choice environment.* For example, if $\mathfrak{X}_{<\infty}$ stands for the collection of all nonempty finite subsets of X, $(X, \mathfrak{X}_{<\infty})$ is a choice environment. This is the environment used by the vast majority of works in the literature. More generally, $(X, \mathbf{k}(X))$ is a choice environment, where X is a topological space and $\mathbf{k}(X)$ stands for the set of all nonempty compact subsets of X.

Given any choice environment $(X, \mathfrak{X})$, a *choice correspondence on* $\mathfrak{X}$ is a set-valued map $C : \mathfrak{X} \rightrightarrows X$ with $C(S) \subseteq S$ for every $S \in \mathfrak{X}$ and $C(S) \neq \varnothing$ for every $S \in \mathfrak{X}_{<\infty}$. We say that C is *single-valued* if $|C(S)| = 1$ for every $S \in \mathfrak{X}_{<\infty}$.

Now take any weak preference structure $(\succsim, \mathbf{R})$ on X. We say that a choice correspondence C on $\mathfrak{X}$ is *rationalized by* $(\succsim, \mathbf{R})$ if

$$C(S) = \bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}), \tag{3}$$

or equivalently,

$$C(S) = \max(\mathbf{MAX}(S, \succsim), \mathrm{tran}(\mathbf{R}|_{\mathrm{MAX}(S, \succsim)})), \tag{4}$$

for every $S \in \mathfrak{X}$. This maintains that a rational agent with a weak preference structure $(\succsim, \mathbf{R})$ makes their choice(s) from a feasible set S by means of a two-step procedure. First, they look for those alternatives in S that are maximal with respect to their core preference $\succsim$. If there is only one such alternative in S, then they choose that alternative. Otherwise, they restrict attention to those alternatives, and evaluate them on the basis of $\mathbf{R}$. Their choice(s) are determined by maximizing $\mathbf{R}$ on $\mathbf{MAX}(S, \succsim)$ in the sense of finding the top-cycle in $\mathbf{MAX}(S, \succsim)$ with respect to $\mathbf{R}$. This top-cycle is the set of all alternatives the agent deems "choosable" in S.

Remark. Following Manzini and Mariotti (2007), we refer to a choice correspondence $S \mapsto \max(\mathbf{MAX}(S, \mathbf{R}_1), \mathbf{R}_2)$, where $\mathbf{R}_1$ and $\mathbf{R}_2$ are binary relations on a finite set X, as a *sequentially rationalized choice procedure.* We note that rationalization by a preference structure is distinct from this form of sequential rationalization. Indeed, for a feasible set

$S \in \mathfrak{X}$, $\mathrm{tran}(\mathbf{R}|_{\mathrm{MAX}(S,\succsim)})$ is not the same relation as $\mathrm{tran}(\mathbf{R})$ in general. Thus, there is no "one" second binary relation used in a choice correspondence rationalized by $(\succsim, \mathbf{R})$.[11]

4.3 Existence of Choice by Preference Structures

A binary relation on a (topological) space X is *continuous* if it is a closed subset of the product space $X \times X$. We thus say that a weak preference structure $(\succsim, \mathbf{R})$ on X is *continuous* if both $\succsim$ and $\mathbf{R}$ are continuous. Our main existence theorem says that $\bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}) \neq \varnothing$ for any such $(\succsim, \mathbf{R})$ and nonempty compact $S \subseteq X$.

Theorem 4.3. *For any space X, the choice correspondence on $\mathbf{k}(X)$ rationalized by a continuous weak preference structure $(\succsim, \mathbf{R})$ on X is nonempty-valued.*

The earliest topological existence theorem for top-cycles is due to Kalai and Schmeidler (1977). While that theorem applies to complete and continuous binary relations only under the hypothesis of antisymmetry, Duggan (2007) showed that the antisymmetry requirement is in fact not needed. In turn, Theorem 4.3 generalizes Duggan's Theorem; the latter obtains from the former simply by setting $\succsim = \triangle_X$. Moreover, under hypotheses of Theorem 4.3, compactness of S does not imply that of $\mathbf{MAX}(S, \succsim)$, so Duggan's Theorem does not apply in the present setting. In the Appendix, we use a direct argument to prove Theorem 4.3.

4.4 Equivalent Preference Structures

In the standard theory, a choice correspondence can be rationalized by at most one complete preference relation (provided that the domain of the correspondence is rich enough). In contrast, the same choice correspondence can be rationalized by many preference structures. We think of such structures as "equivalent" from the perspective of choice.

Definition. Given any choice environment $(X, \mathfrak{X})$, two weak preference structures $(\succsim, \mathbf{R})$ and $(\succsim', \mathbf{R}')$ on X are said to be *equivalent* if

$$\bigcirc(\mathbf{MAX}(S, \succsim), \mathbf{R}) = \bigcirc(\mathbf{MAX}(S, \succsim'), \mathbf{R}')$$

[11] When X is finite and $S, T \in \mathfrak{X}$, we have $C(S) \cap C(T) \subseteq C(S \cup T)$ for any sequentially rationalized choice correspondence C on $\mathfrak{X}$ (García-Sanz and Alcantud (2015)). Now, let $X := \{x_1, ..., x_5\}$, $\succsim := \triangle_X \sqcup \{(x_3, x_5), (x_4, x_2)\}$, and consider the total binary relation $\mathbf{R}$ on X with $\mathbf{R}^> := \{(x_3, x_5), (x_4, x_2), (x_3, x_1), (x_4, x_1)\}$. Then, $(\succsim, \mathbf{R})$ is a preference structure on X. But for $S := \{x_1, x_2, x_3\}$, $T := \{x_1, x_4, x_5\}$, and C defined by (3), $x_1 \in (C(S) \cap C(T)) \backslash C(S \cup T)$.

for every $S \in \mathfrak{X}$; we denote this situation by writing $(\succsim, \mathbf{R}) \cong (\succsim', \mathbf{R}')$.

In Section 3.3, we noted that there is a "natural" way of pruning a weak preference structure to make it a proper preference structure that is indistinguishable from the former in terms of choice theory. Indeed, any weak preference structure is equivalent to its proper representation:

Proposition 4.5. *Let $(X, \mathfrak{X})$ be a choice environment, and $(\succsim, \mathbf{R})$ a weak preference structure on X. Then, $(\succsim, \mathbf{R}) \cong (\succsim, \mathbf{R}_{\succsim})$ where $\mathbf{R}_{\succsim}$ is defined by (1).*

Proof. Take an arbitrary $S \in \mathfrak{X}$. There is nothing to prove if there is no $\succsim$-maximal element in S. Assume otherwise, and take any $x, y \in \mathbf{MAX}(S, \succsim)$. If $x \sim y$ we have $x \, \mathbf{R}^{=} \, y$ and $x \, \mathbf{R}_{\succsim}^{=} \, y$ (because both $\mathbf{R}$ and $\mathbf{R}_{\succsim}$ are superrelations of $\succsim$). If $(x, y) \in \mathrm{Inc}(\succsim)$, then $x \, \mathbf{R} \, y$ iff $x \, \mathbf{R}_{\succsim} \, y$ by definition of $\mathbf{R}_{\succsim}$. Thus, the restrictions of $\mathbf{R}$ and $\mathbf{R}_{\succsim}$ to $\mathbf{MAX}(S, \succsim)$ are the same. By Proposition 4.1, therefore, $(\succsim, \mathbf{R}) \cong (\succsim, \mathbf{R}_{\succsim})$. ■

The takeaway is: A choice correspondence is rationalizable by a weak preference structure if, and only if, it is rationalizable by a proper preference structure.

4.5 Examples

The following examples are meant to illustrate the breadth of rationalization by preference structures. Unless otherwise is explicitly stated, $(X, \mathfrak{X})$ stands below for an arbitrarily fixed choice environment.

Example 4.1. (*The Rational Choice Model*) For any complete preorder $\succsim$ on X, the choice correspondence C on $\mathfrak{X}$ rationalized by the transitive preference structure $(\succsim, \succsim)$ satisfies $C(S) = \max(S, \succsim)$ for every $S \in \mathfrak{X}$. Thus, the choice theory based on preference structures generalizes the standard theory.

Example 4.2. (*The Top-Cycle Choice Rule*) For any complete binary relation $\mathbf{R}$ on X, the choice correspondence C on $\mathfrak{X}$ rationalized by the preference structure $(\triangle_X, \mathbf{R})$ satisfies $C(S) = \bigcirc(S, \mathbf{R})$ for every $S \in \mathfrak{X}$. Thus, the choice theory based on preference structures generalizes the theory of top-cycle choice rules that are commonly used in the theory of social choice and tournaments (cf. Kalai and Schmeidler (1977), Schwartz (1986), Laslier (1997), and Ehlers and Sprumont (2008)).

Example 4.3. (*The Undominated Choice Rule*) Let $\succsim$ be a preorder on X. Then, the choice correspondence C on $\mathfrak{X}$ rationalized by the preference structure $(\succsim, \succsim \sqcup$

Inc($\succsim$)) satisfies $C(S) = \mathbf{MAX}(S, \succsim)$ for every $S \in \mathfrak{X}$. Thus, the choice theory based on preference structures generalizes the choice theory that is based on incomplete (but transitive) preference relations (cf. Eliaz and Ok (2006)).

Example 4.4. (*Pareto Refinement of Majority Voting*) Let $\mathcal{P}$ and $\mathcal{P}_{\mathrm{maj}}$ be defined as in Example 3.5. Then, the choice correspondence C on $\mathfrak{X}$ rationalized by the preference structure $(\bigcap \mathcal{P}, \mathcal{P}_{\mathrm{maj}})$ assigns to any feasible set $S \in \mathfrak{X}$ those Pareto optimal outcomes in S that maximize the transitive closure of the majority voting rule on S.

Example 4.5. (*Transitive Preference Structures*) As shown in Section 4.2, a choice correspondence rationalized by a preference structure need not be a sequential choice procedure (in the sense of Manzini and Mariotti (2007)). However, the situation is different in the transitive case. To wit, let C be the choice correspondence on $\mathfrak{X}$ rationalized by a transitive weak preference structure on X. Then, $C(S) = \max(\mathbf{MAX}(S, \succsim), \mathbf{R})$ for every $S \in \mathfrak{X}$. This sits square with the interpretation that $\succsim$ is the "sure" preferences of a person, and $\mathbf{R}$ corresponds to the rational (complete) preferences of a consultant.

Remark. Suppose X is a finite set. Then, $C = \max(\cdot, \mathbf{R})$ if $(\succsim, \mathbf{R})$ is a transitive *proper* preference structure. This is not true if $(\succsim, \mathbf{R})$ is a transitive *weak* preference structure, however. In that case, $C \subseteq \max(\cdot, \mathbf{R})$, but easy examples show that the converse containment need not hold.

Example 4.6. (*The Constant Threshold Choice Model*) Take any $u : X \to \mathbb{R}$ and $\varepsilon \geq 0$. Define the binary relation $\mathbf{R}$ on X as $x \mathbf{R} y$ iff $u(x) \geq u(y) - \varepsilon$. Consider first the preorder $\succsim'$ on X with $x \succsim' y$ iff either $x = y$ or $u(x) > u(y)$. Then, $(\succsim', \mathbf{R})$ is a weak preference structure on X, and the choice correspondence C on $\mathfrak{X}$ rationalized by $(\succsim', \mathbf{R})$ is the rational choice model: $C(S) = \arg\max\{u(x) : x \in S\}$ for every $S \in \mathfrak{X}$. Next, consider the preorder $\succsim$ on X with $x \succsim y$ iff either $x = y$ or $u(x) > u(y) + \varepsilon$. Then, $(\succsim, \mathbf{R})$ is a preference structure on X (Example 3.7), and the choice correspondence C on $\mathfrak{X}$ rationalized by $(\succsim, \mathbf{R})$ satisfies

$$C(S) = \{x \in S : \sup u(S) - u(x) \leq \varepsilon\}$$

for every $S \in \mathfrak{X}$, that is, it is a *constant threshold choice model* (in the sense of Luce (1956)). Conclusion: Every constant threshold choice model is rationalized by a preference structure.

4.6 On the Predictive Power of the Model

A general model of rational choice must exhibit some behavioral restrictions; otherwise, it explains everything, and predicts nothing. In this section we thus look into some of the behavioral implications of our choice theory. We begin with a simple example that shows that not every choice correspondence is compatible with this theory.

Example 4.7. Put $X := \{x, y, z\}$, and take any choice correspondence C on $\mathfrak{X}_{<\infty}$ with $x \in C\{x, y\}$ and $\{y\} = C\{x, y, z\}$. To derive a contradiction, suppose C is rationalized by a preference structure $(\succsim, \mathbf{R})$ on X. As $y \in C\{x, y, z\}$, y is $\succsim$-maximal in $\{x, y\}$, so $x \in C\{x, y\}$ implies $x \mathbf{R} y$ (Proposition 4.1). Since x does not belong to $C\{x, y, z\}$ but y does, therefore, x is not $\succsim$-maximal in X. As $y \succ x$ cannot hold (because $x \in C\{x, y\}$), we thus have $z \succ x$. Then, $y \succ z$ cannot hold (because $\succ$ is transitive). It follows that z is $\succsim$-maximal in X, and hence $y \mathbf{R}^{>} z$ (because $\{y\} = C\{x, y, z\}$). Thus, $z \succ x \mathbf{R} y$ and $y \mathbf{R}^{>} z$, which contradicts $\succsim$-transitivity of $\mathbf{R}$.

Weak Axiom of Revealed Preference. Recall that $\mathfrak{X}_{<\infty}$ stands for the family of all nonempty finite subsets of X. A choice correspondence C on $\mathfrak{X}_{<\infty}$ satisfies *Property* α if $C(T) \cap S \subseteq C(S)$ for every $S, T \in \mathfrak{X}_{<\infty}$ with $S \subseteq T$, and it satisfies *Property* β if for every $S, T \in \mathfrak{X}_{<\infty}$ with $S \subseteq T$, and $x, y \in C(S)$ with $x \in C(T)$, we have $y \in C(T)$. We say that C obeys the *Weak Axiom of Revealed Preference* (*WARP*) if it satisfies both of these properties; this is a fundamental tenet of rational choice. It is well-known that C satisfies WARP iff there is a complete preference relation $\succsim$ on X with $C(S) = \max(S, \succsim)$ for all $S \in \mathfrak{X}_{<\infty}$.

A choice correspondence rationalized by a preference structure need not satisfy either Property α or Property β. After all, choice correspondences of the form considered in Examples 4.2 and 4.3 are well-known to fail Properties α and β, respectively. In fact, Example 4.3 shows that even a choice correspondence on $\mathfrak{X}_{<\infty}$ that is rationalized by transitive weak preference structure may violate Property β.

Single-Valued Choice Correspondences. While rationalization by a preference structure ensures, in general, neither Property α nor Property β, it turns out that it reduces to the standard notion of rationalization in the context of *single-valued* choice correspondences.

Proposition 4.6. *Let X be any nonempty set, and C a single-valued choice correspondence on $\mathfrak{X}_{<\infty}$. Then, C is rationalized by a weak preference structure if, and only if, it satisfies WARP.*

Proof. The "if" part is straightforward. Conversely, suppose C is rationalized by a weak preference structure $(\succsim, \mathbf{R})$ on X. Take any $x \in X$ and $S, T \in \mathfrak{X}_{<\infty}$ with $S \subseteq T$ and $\{x\} = C(T)$. Then, $x \in \mathbf{MAX}(T, \succsim)$ and $x \ \mathbf{R} \ \mathbf{MAX}(T, \succsim)$. To derive a contradiction, suppose $x \notin C(S)$. Since $S \subseteq T$, we have $x \in \mathbf{MAX}(S, \succsim)$ so there must then exist a $y \in \mathbf{MAX}(S, \succsim)$ with $y \ \mathbf{R}^{>} \ x$. Since $y \notin C(T)$, therefore, y cannot be $\succsim$-maximal in T. Since T is finite, this means that $z \succ y$ for some $z \in \mathbf{MAX}(T, \succsim)$. Then, $z \ \mathbf{R} \ x$ by $\succsim$-transitivity of $\mathbf{R}$. By the choice of x, this is possible only if $z = x$, so this implies $x \succ y$, contradicting $\succsim$-maximality of y in S. Conclusion: C satisfies Property α. As for single-valued choice correspondence WARP is equivalent to Property α, we are done. ■

Proposition 4.6 is another demonstration of the predictive power of rationalization by preference structures. It shows that our choice theory reduces to the standard theory of rational choice in the context of single-valued choice correspondences on finite choice problems. (Thus, for instance, the rational shortlisting models of Manzini and Mariotti (2007), Au and Kawai (2011), and Cherepanov et al. (2013), as well as the attention/competition filter model of Masatlioglu, Nakajima and Ozbay (2012), are not captured by this theory.) This is not surprising. The main goal of the model of preference structures is to capture behavioral traits such as indecisiveness and cyclical choices, and as such, the choice theory that it induces is primed to make many-valued predictions.

The Condorcet Criterion. Let $(X, \mathfrak{X})$ be a choice environment. A choice correspondence C on $\mathfrak{X}$ is said to satisfy the *Condorcet Criterion* if for every $S \in \mathfrak{X}$ and $x \in S$,

$$x \in C\{x, y\} \text{ for every } y \in S \quad \text{imply} \quad x \in C(S).$$

The choice behavior that is rationalizable by a preference structure is sure to be consistent with this property.

Proposition 4.7. *Let C be a choice correspondence on $\mathfrak{X}$ rationalized by a weak preference structure $(\succsim, \mathbf{R})$ on X. Then, C satisfies the Condorcet Criterion.*

Proof. Take any $S \in \mathfrak{X}$ and $x \in S$ with $x \in C\{x, y\}$ for all $y \in S$. Then, $x \in \mathbf{MAX}(S, \succsim)$ so, given that $C(S)$ is the top-cycle in $\mathbf{MAX}(S, \succsim)$ with respect to $\mathbf{R}$, if x did not belong to $C(S)$, we would have $y \ \mathbf{R}^{>} \ x$ for some $y \in \mathbf{MAX}(S, \succsim)$. But then, for this y, we would have $\{y\} = C\{x, y\}$, a contradiction. ■

Thus, despite being based on incomplete and nontransitive relations, the choice model induced by preference structures respects the Condorcet Criterion, thereby imposing nontrivial restrictions on observed choice behavior.

Proposition 4.7 distinguishes choice correspondences rationalized by preference structures from boundedly rational choice correspondences. For instance, we see readily that the reference-dependent choice model of Ok, Ortoleva and Riella (2015), or the model of choice with limited consideration, introduced by Lleras et al. (2017), are distinct from the present choice model as they do not satisfy the Condorcet Criterion. For the same reason, justifiable choice correspondences of Heller (2012) and Costa, Ramos and Riella (2020) are, in general, not rationalizable by a weak preference structure.

Similarly, not all threshold choice models, let alone all satisficing rules, are rationalizable by preference structures. For example, consider the choice correspondence C on $\mathfrak{X}$ defined by

$$C(S) = \{x \in S : \sup u(S) - u(x) \leq \varepsilon(S)\},$$

where u is any real map on X and $\varepsilon : \mathfrak{X} \to \mathbb{R}_+$ is a function such that $\varepsilon(A) \leq \varepsilon(B)$ for every $A, B \in \mathfrak{X}$ with $A \subseteq B$. (Where X is finite, Frick (2016) has provided an axiomatization of such choice correspondences, and called them *monotone threshold choice models.*) Such a model need not be rationalizable by a preference structure.

Other Behavioral Consistency Properties. Suppose C is a rationalizable choice correspondence on $\mathfrak{X}$ in the classical sense, that is, $C = \max(\cdot, \succsim)$ for some complete preorder $\succsim$ on X. Let z be a choice from a feasible set S by a decision maker whose choice behavior is modeled by this choice correspondence. If this agent is instead offered the feasible set $S \cup \{x\}$ where x is a new alternative at least as good as z, she would surely deem x choosable from this set: $x \in C(S \cup \{x\})$. It is in this sense that C is *monotonic* with respect to $\succsim$.

Let C be the choice correspondence rationalized by some preference structure $(\succsim, \mathbf{R})$ on X. Suppose $z \in C(S)$ for some $S \in \mathfrak{X}$. Then, if a rational agent has no doubt in their mind that x is a better alternative than z, that is, $x \succsim z$, one would expect they view x as choosable from $S \cup \{x\}$. The following proposition shows that C possesses this property indeed.

Proposition 4.8. *Let C be a choice correspondence on $\mathfrak{X}$ rationalized by a weak preference structure $(\succsim, \mathbf{R})$ on X. Then, for any $S \in \mathfrak{X}$,*

$$x \succsim z \in C(S) \quad \textit{implies} \quad x \in C(S \cup \{x\}).$$

Let us now ask the same question with respect to the revealed preference relation $\mathbf{R}$. That is, suppose $z \in C(S)$ for some $S \in \mathfrak{X}$, and that we have observed the agent choose

x over z. Would this agent necessarily deem x choosable from $S \cup \{x\}$? This is less clear-cut. The decision maker may have chosen x over z with serious difficulty, perhaps referring to the preferences of another individual. Thus, it is possible that some alternatives in S may dominate x, but not z, with respect to their core preferences, and this may cause x be not chosen from $S \cup \{x\}$ even though z is deemed choosable from S. This may indeed be the case. However, if z is the *only* choice from S, C still acts monotonically with respect to $\mathbf{R}$.

Proposition 4.9. *Let C be a choice correspondence on $\mathfrak{X}$ rationalized by a preference structure $(\succsim, \mathbf{R})$ on X. Then, for any $S \in \mathfrak{X}_{<\infty}$,*

$$x \mathbf{R} z \text{ and } \{z\} = C(S) \quad \text{imply} \quad x \in C(S \cup \{x\}).$$

5 Observability of Preference Structures

Rational choice theory is built on the hypothesis that the choice behavior of a rational individual is the result of the maximization of a *unique* complete preorder. In the context of any choice environment, every complete preference relation gives rise to a unique rationalizable choice model, and every rationalizable choice model induces a unique preference relation (that arises from pairwise choice problems). While trivial, this duality is an essential aspect of rational choice theory.

In this section, we investigate the extent to which such a duality exists for our choice model. That is, we examine the relation between two preference structures that happen to rationalize the same choice correspondence. This amounts to understanding exactly how two *equivalent* preference structures $(\succsim, \mathbf{R})$ and $(\succsim', \mathbf{R}')$ relate to each other. Of course, the exact analogue of the situation in rational choice theory would be to have $\succsim = \succsim'$ and $\mathbf{R} = \mathbf{R}'$. The second of these equations is indeed correct (provided we work with *proper* preference structures), but mainly because different preorders with the same asymmetric part would declare the same elements as maximal in all feasible sets, the first equation is, in general, false. (For instance, $(\triangle_X, X \times X)$ and $(X \times X, X \times X)$ rationalize the same choice correspondence.) However, we will show below that one can always identify "the" largest preference structure – this is the one whose core preference exhibits the least amount of incompleteness – that rationalizes a choice correspondence. Thus, the present choice model too exhibits a useful, and this time entirely nontrivial, duality. Every preference structure gives rise to a unique rationalizable choice model (in the context of a suitably general choice environment), and conversely, every choice model that is rationalizable by a preference structure induces a unique largest preference structure.

Revealed Preferences. Let $(X, \mathfrak{X})$ be a choice environment, and C a choice correspondence on $\mathfrak{X}$. We define the *revealed preference relation* induced by C as the complete binary relation $\mathbf{R}_C$ on X with

$$x \ \mathbf{R}_C \ y \quad \text{iff} \quad x \in C\{x, y\}.$$

In words, $x \ \mathbf{R}_C \ y$ means the agent (with choice correspondence C) would choose x over y when comparing these two alternatives alone. The following elementary observation highlights the importance of this relation.

Lemma 5.1. *Let $(X, \mathfrak{X})$ be a choice environment, and C a choice correspondence on $\mathfrak{X}$. If C is rationalized by a preference structure $(\succsim, \mathbf{R})$ on X, then $\mathbf{R} = \mathbf{R}_C$.*

Proof. For any $x, y \in X$, setting $S = \{x, y\}$ in (3) yields $\{x\} = C\{x, y\}$ iff $x \ \mathbf{R}^{>} \ y$, and $\{x, y\} = C\{x, y\}$ iff $x \ \mathbf{R}^{=} \ y$. Thus: $\mathbf{R} = \mathbf{R}_C$. ■

Revealed preferences of an individual whose choices are rationalized by a preference structure are thus uniquely identified from their binary choice decisions. This shows that the interpretation of preference structures we outlined in Section 3.1 is duly consistent with the choice theory introduced in Section 4.

We should emphasize that Lemma 5.1 is not valid for *weak* preference structures. In that case $(\succsim, \mathbf{R}) \cong (\succsim, \mathbf{R}_{\succsim})$ by Proposition 4.1, but $\mathbf{R}$ and $\mathbf{R}_{\succsim}$ may well be distinct. Thus, at least from the perspective of preference identification, proper preference structures appear to be superior to weak preference structures.

Revealed Core Preferences. Unlike its revealed part, the core part of a preference structure is not observable by an outside observer, so it is particularly important to understand which sorts of preorders on X rationalize a given choice correspondence C on $\mathfrak{X}$ when coupled with $\mathbf{R}_C$. We denote the set of all such preorders by $\mathbb{P}(C)$, that is,

$$\mathbb{P}(C) := \{\succsim \, : \, (\succsim, \mathbf{R}_C) \text{ is a preference structure on } X \text{ that rationalizes } C\}.$$

Thus: $\mathbb{P}(C) \neq \varnothing$ *iff C is rationalizable by a preference structure*. Moreover, for any $\succsim$ and $\succsim'$ in $\mathbb{P}(C)$, we have $(\succsim, \mathbf{R}_C) \cong (\succsim', \mathbf{R}_C)$, that is, choice-theoretically, there is no difference between $\succsim$ and $\succsim'$.

There is a natural way of partially ordering all preorders on X on the basis of their completeness. For any two such preorders $\succsim$ and $\succsim'$, we say that $\succsim$ is *more complete* *than* $\succsim'$ if the former preorder extends the latter. We denote this partial order, as well as any restriction of it to a given set of preorders on X, by $\sqsupseteq$. (In particular, for any $\succsim$ and $\succsim'$ in $\mathbb{P}(C)$, we have $\succsim \sqsupseteq \succsim'$ iff $\succsim$ extends $\succsim'$.) In what follows, we consider $\mathbb{P}(C)$ as a poset relative to $\sqsupseteq$. Even at this level of generality, this poset has a remarkable structure:

Theorem 5.2. *Let $(X, \mathfrak{X})$ be a choice environment, and C a choice correspondence on $\mathfrak{X}$. If C is rationalized by at least one preference structure on X, then $\mathbb{P}(C)$ is a complete $\vee$-semilattice.*[12]

This result, whose proof is somewhat involved, is by no means a technical observation. It provides a clear insight about those core preferences that rationalize a given choice correspondence C when coupled with revealed preferences. Apparently, any collection of such core preference relations can be combined to get another, more decisive, core preference that still rationalizes C when paired with $\mathbf{R}_C$. In particular, there is a *most decisive* core preference on X. (We will compute this preference shortly.)

Remark. $\mathbb{P}(C)$ need not be a $\wedge$-semilattice under the hypotheses of Theorem 5.2. The proof is available upon request.

The Largest Revealed Core Preference. The revealed preference relation induced by C arises only through pairwise comparisons of alternatives. It does not say much about the choosability of one attribute over another across all feasible sets. This task would be handled by those preorders $\succsim$ that are compatible with $\mathbf{R}_C$. Indeed, according to our interpretation of $(\succsim, \mathbf{R}_C)$, $x \succ y$ means that the agent prefers x over y "obviously," so it stands to reason that she would never choose y in any situation in which x is feasible. This prompts looking at the asymmetric binary relation $\succ_C$ on X with

$$x \succ_C y \quad \text{iff} \quad y \notin C(S) \text{ for every } S \in \mathfrak{X} \text{ with } x \in S.$$

We refer to $\succ_C$ as the *revealed core dominance* induced by C.

It is also possible that x and y are "obviously" equally appealing for the decision maker. From the vantage point of choice theory, this means that replacing x with y in any choice problem does not alter the choice behavior of the agent, apart from

[12] A poset $(A, \trianglerighteq)$ is said to be a *complete $\vee$-semilattice* if for every nonempty $B \subseteq A$, the $\trianglerighteq$-minimum of the set $\{a \in A : a \trianglerighteq B\}$ exists; *complete $\wedge$-semilattices* are defined dually. If $(A, \trianglerighteq)$ is both a complete $\vee$-semilattice and a complete $\wedge$-semilattice, we say that it is a *complete lattice*.

the replacement of x with y. That is, if x is deemed choosable (or unchoosable) in a menu, replacing x with y in that menu would yield a menu in which y is choosable (or, respectively, unchoosable). In addition, the choosability status of any other alternative in the two menus remains the same. We are thus led to define the binary relation $\sim_C$ on X by $x \sim_C y$ iff

$$x \in C(S \cup \{x\}) \text{ iff } y \in C(S \cup \{y\})$$

and

$$z \in C(S \cup \{x\}) \text{ iff } z \in C(S \cup \{y\})$$

for every $S \in \mathfrak{X}$ and every $z \in S$. Put succinctly, $x \sim_C y$ means that x and y are perfect substitutes in that replacing one for the other does not change the choice behavior of the agent in any choice situation. This relation, which we borrow from Riberio and Riella (2017), is called the *revealed core indifference* induced by C. It is a symmetric relation disjoint from $\succ_C$.

Finally, we define $\succsim_C$ as the union of the relations $\succ_C$ and $\sim_C$, and refer to it as the *revealed core preference relation* induced by C. In general, $\succsim_C$ is a subrelation of $\mathbf{R}_C$. In fact, not only is $(\succsim_C, \mathbf{R}_C)$ is a preference structure on X that rationalizes C, but it is the largest such structure. Put differently, $\succsim_C$ is the top element of the $\vee$-semilattice $\mathbb{P}(C)$.

Theorem 5.3. *Let $(X, \mathfrak{X})$ be any choice environment, and C a nonempty-valued choice correspondence on $\mathfrak{X}$. If C is rationalized by at least one preference structure on X, then $\bigvee \mathbb{P}(C) = \succsim_C$.*

This result characterizes the most decisive core preference relation compatible with a choice correspondence. While the core part of a preference structure is, in general, not observable and non-unique, we can still elicit this part, in its most decisive form, from one's choice behavior.

Taking stock: Given any choice correspondence C that is rationalized by a preference structure, we can identify $(\succsim_C, \mathbf{R}_C)$ as the preference structure of the involved decision-maker. Insofar as the values of C are observable across all choice problems, $(\succsim_C, \mathbf{R}_C)$ is perfectly observable, and further, the choice correspondence that $(\succsim_C, \mathbf{R}_C)$ rationalizes is precisely C. We can be sure that $\mathbf{R}_C$ corresponds to the actual revealed preference of the agent. While $\succsim_C$ may not be identical to their actual core preference, it is the most decisive version of that preference. Indeed, it is "equivalent" to it insofar as the choices of the agent is concerned. And it is the most complete of all preference relations that is equivalent to their actual core preference in this sense. This identifies in exactly how we may recover one's preference structure from their observed choice behavior. So long as we pick the revealed core preference and revealed preference relations induced by C as "representative," then the models $(\succsim_C, \mathbf{R}_C)$ and C stand dual to each other.

Example 5.1. (*Revealed Preferences with Imperfect Discrimination*) Take any integer $n \geq 2$, and let $u : \mathbb{R}^n \to \mathbb{R}$ be a continuous surjection. Pick any $\varepsilon > 0$, and consider the preference structure $(\succsim, \mathbf{R})$ on $\mathbb{R}^n$ where $x \mathbf{R} y$ iff $u(x) \geq u(y) - \varepsilon$, and $x \succsim y$ iff either $x = y$ or $u(x) > u(y) + \varepsilon$. Let C be the choice correspondence on $\mathbf{k}(\mathbb{R}^n)$ rationalized by $(\succsim, \mathbf{R})$. Using the characterization of C given in Example 4.5, one can show that

$$x \succsim_C y \quad \text{iff} \quad \text{either } u(x) = u(y) \text{ or } u(x) > u(y) + \varepsilon.$$

So, in this case, $\succ = \succ_C$, but (as u cannot be injective), we have $\succsim \neq \succsim_C$.

Remark. For expositional purposes, we have not stated Theorem 5.3 above in its strongest form. It turns out that the nonempty-valuedness hypothesis can be omitted in the statement of this theorem, but at the cost of lengthening the proof significantly. In view of Theorem 4.3, however, this hypothesis is largely inconsequential for applications.

Remark. The ordering of $\mathbb{P}(C)$ used in Theorem 5.2 exhibits a somewhat unnatural asymmetry: While any two rationalizing core preferences can be combined to get a more decisive such preference, we cannot, in general, find a less decisive such preference. It is worth noting that in most applications this issue does not arise. Indeed, if X is a topological space, and C is the choice correspondence on $\mathbf{k}(X)$ rationalized by a continuous preference structure $(\succsim, \mathbf{R})$ on X, then $\mathbb{P}(C)$ is a complete lattice. The proof of this result is avaliable from the authors upon request.[13]

6 Conclusion

In this paper, we modeled the preferences of an economic agent on a set X of choice prospects by means of two binary relations, $\succsim$ and $\mathbf{R}$, on X. The first of these is transitive (but not necessarily complete), and captures those rankings that are (subjectively) "obvious/easy." $\mathbf{R}$ is complete (but not necessarily transitive), and it

[13] It may be of interest to see what the *least decisive* core preference that rationalizes C when combined with $\mathbf{R}_C$, namely, $\bigwedge \mathbb{P}(C)$ looks like. This relation is found as $\text{tran}(\rhd) \cup \triangle_X$, where $\rhd$ is a binary relation on X defined by $x \rhd y$ iff there exist a $k \in \mathbb{N}$ and $z_1, \ldots, z_k$ in X such that (i) $y \, \mathbf{R}_C \, z_1 \, \mathbf{R}_C \cdots \mathbf{R}_C \, z_k \, \mathbf{R}_C \, x$, (ii) $y \in C\{y, z_1, \ldots, z_k\}$, (iii) $\{x, z_1, \ldots, z_k\} \subseteq C(S)$ for some $S \in \mathbf{k}(X)$ with $y \in S$, and (iv) $y \notin C\{x, y, z_1, \ldots, z_k\}$. Again, the proof is available upon request.

arises from the observed choices of in the context of pairwise choice problems. We posited that $\succsim$ and $\mathbf{R}$ are consistent with each other (as they arise from the preferences of the same agent) in the sense that (i) $\mathbf{R}$ is an extension of $\succsim$, and (ii) $\mathbf{R}$ is transitive with respect to $\succsim$. This way one captures many preference models (where the decision-making unit may be a group of individuals). Among these are the models of incomplete preferences, preferences with imperfect ability of discrimination, regret preferences, and preferences completed by the recommendations of a consultant.

Our main goal was to develop a model of choice behavior that arises from preference structures. We did this by using a two-step procedure. In the first step of this procedure $\succsim$ is maximized, and in the second $\mathbf{R}$ is maximized (in the sense of finding the top-cycle over the set of all $\succsim$-maximal elements). This led to a fairly rich theory that generalizes the classical rational choice theory. The explanatory power of this alternate theory is obviously superior to the classical one, but it also has a good deal of predictive power. In particular, this model is menu-independent, and it satisfies the Condorcet Criterion. Moreover, this model is on the same footing with the standard model in terms of existence. It also allows us identify (not fully, but to a great extent) one's core, welfare-relevant preferences from their observed choice behavior.

It will be interesting to revisit the classical consumer theory, as well as theories of decision making under risk and uncertainty, this time using preference structures instead of preference relations. Similarly, one should investigate how (ordinal) game theory would look like when we model the preferences of the players through preference structures. It is also of interest to see how one may model time preferences through preference structures, and then revisit the theory of optimal saving. These, and numerous other avenues remain to be explored.

APPENDIX: Proofs

This appendix contains the proofs of the results that were omitted in the body of the text.

Proof of Proposition 3.1

Let X be a nonempty set, and $(\succsim, \mathbf{R})$ a weak preference structure on X. That $\mathbf{R}_{\succsim}$ is a completion of $\succsim$ follows readily from the definition of $\mathbf{R}_{\succsim}$, so we need only to prove that $\mathbf{R}$ is $\succsim$-transitive. To this end, take any x, y and z in X such that $x \ \mathbf{R}_{\succsim} \ y \succsim z$. Notice that $z \succ x$ cannot hold, because otherwise $y \succ x$ (by transitivity of $\succsim$), and hence $y \ (\mathbf{R}_{\succsim})^{>} \ x$ (because $\mathbf{R}_{\succsim}$ is an extension of $\succsim$), a contradiction. Thus: Either $x \succsim z$ or $(x, z) \in \mathrm{Inc}(\succsim)$. In the former case, we have $x \ \mathbf{R}_{\succsim} \ z$ by definition of $\mathbf{R}_{\succsim}$, so we are done. Similarly, if $x \succsim y$, then $x \ \mathbf{R}_{\succsim} \ z$ because $\succsim$ is transitive and $\succsim \subseteq \mathbf{R}_{\succsim}$. So, assume that $(x, z) \in \mathrm{Inc}(\succsim)$ and $x \succsim y$ is false. Since $x \ \mathbf{R}_{\succsim} \ y$, the latter statement and the definition of $\mathbf{R}_{\succsim}$ imply that $x \ \mathbf{R}$

y. Then, $x \mathbf{R} y \succsim z$, and hence $x \mathbf{R} z$ by $\succsim$-transitivity of $\mathbf{R}$. It follows that $(x,z) \in \text{Inc}(\succsim)$ and $x \mathbf{R} z$, that is, $x \mathbf{R}_{\succsim} z$, as we sought. As we can similarly show that $x \succsim y \mathbf{R}_{\succsim} z$ implies $x \mathbf{R}_{\succsim} z$, we conclude that $(\succsim, \mathbf{R}_{\succsim})$ is a preference structure on X.

Proof of Corollary 3.2

Let X be a nonempty set, and $(\succsim, \mathbf{R})$ a transitive weak preference structure on X. In view of Proposition 3.1, we only need to prove that $\mathbf{R}_{\succsim}$ is quasitransitive. We will in fact prove something stronger than this below.

First, recall that a binary relation $\mathbf{S}$ is said to be **Suzumura consistent** if $x \text{ tran}(\mathbf{S})\ y$ implies not $y \mathbf{S}^{>} x$ for every $x, y \in X$. Second, note that it is easy to verify that if $(\succsim, \mathbf{S})$ is a preference structure such that $\text{Inc}(\succsim) \cap \mathbf{S}$ is Suzumura consistent, then $\mathbf{S}$ is quasitransitive. Thus, Proposition 3.1 will be proved if we can show that $\text{Inc}(\succsim) \cap \mathbf{R}_{\succsim}$ is Suzumura consistent. To this end, put $\mathbf{T} := \text{Inc}(\succsim) \cap \text{ tran}(\mathbf{R}_{\succsim})$, and note that

$$\mathbf{T} = \text{Inc}(\succsim) \cap \mathbf{R}_{\succsim}. \tag{5}$$

(Indeed, for any $(x, y) \in \mathbf{T}$, there exist finitely many $x_1, ..., x_k \in X$ such that $x \mathbf{R}_{\succsim} x_1 \mathbf{R}_{\succsim} \cdots \mathbf{R}_{\succsim} x_k \mathbf{R}_{\succsim} y$. Since $\mathbf{R}_{\succsim} \subseteq \succsim \cup \mathbf{R} \subseteq \mathbf{R}$, we then have $x \mathbf{R} x_1 \mathbf{R} \cdots \mathbf{R} x_k \mathbf{R} y$, so given that $\mathbf{R}$ is transitive, we find $x \mathbf{R} y$. As $(x, y) \in \text{Inc}(\succsim)$, this means $x \mathbf{R}_{\succsim} y$. Thus, $\mathbf{T} \subseteq \text{Inc}(\succsim) \cap \mathbf{R}_{\succsim}$, while the converse containment is trivially true.) Now, to derive a contradiction, suppose $\text{Inc}(\succsim) \cap \mathbf{R}_{\succsim}$ is not Suzumura consistent. Then, by (5), $\mathbf{T}$ is not Suzumura consistent, so there exists a $k \in \mathbb{N}$ and $x_1, \ldots, x_k$ in X such that $x_1 \mathbf{T} \cdots \mathbf{T} x_k \mathbf{T} x_1$ with at least one of these $\mathbf{T}$ holding strictly. By relabeling if necessary, we may assume that $x_k \mathbf{T}^{>} x_1$. Then, $x_1 \mathbf{R}_{\succsim} \cdots \mathbf{R}_{\succsim} x_k \mathbf{T}^{>} x_1$, that is, $x_1 \text{ tran}(\mathbf{R}_{\succsim})\, x_k\, (\mathbf{R}_{\succsim})^{>} x_1$. Since x_1 and x_k are not $\succsim$-comparable, it follows that $x_1 \mathbf{T} x_k \mathbf{T}^{>} x_1$, a contradiction.

Proof of Theorem 3.3

The proof of the "if" part of the assertion is straightforward, so we focus only on its "only if" part. Let $(\succsim, \mathbf{R})$ be a preference structure on X. (The proof for a weak preference structure is given analogously.) Put $\mathbf{T} := \mathbf{R} \setminus \succsim$. We may assume that $\mathbf{T}$ is nonempty, for otherwise there is nothing to prove.

Claim. For every $(x, y) \in \mathbf{T}$, there is a preorder $\succsim_{(x,y)}$ on X such that (i) $\succsim_{(x,y)}$ extends $\succsim$, (ii) $\succsim_{(x,y)} \subseteq \mathbf{R}$, and (iii) $x \succsim_{(x,y)} y$.[14]

Proof of Claim. Fix any $(x, y) \in \mathbf{T}$, and define

$$\succsim_{(x,y)} \;:= \succsim \cup\, (x^{\uparrow,\succsim} \times y^{\downarrow,\succsim}).$$

[14]When $(\succsim, \mathbf{R})$ is a weak preference structure, we verify a weaker property of (i), namely, that $\succsim_{(x,y)}$ includes $\succsim$, which trivially follows by the construction of $\succsim_{(x,y)}$.

That $\succsim_{(x,y)}$ is a preorder with $x \succsim_{(x,y)} y$ is verified routinely. To prove (ii), take any $a, b \in X$ with $a \succsim_{(x,y)} b$. If (a, b) does not belong to $\mathbf{R}$, then it does not belong to $\succsim$ either (because $\mathbf{R}$ is a superrelation of $\succsim$). In that case, then, (a, b) belongs to $x^{\uparrow,\succsim} \times y^{\downarrow,\succsim}$, so we have $a \succsim x$ $\mathbf{R}$ $y \succsim b$, which, by $\succsim$-transitivity of $\mathbf{R}$, implies a $\mathbf{R}$ b, a contradiction. We thus conclude that $\succsim_{(x,y)} \subseteq \mathbf{R}$. It remains to check that $\succsim_{(x,y)}$ extends $\succsim$. Obviously, $\succsim$ is a subrelation of $\succsim_{(x,y)}$. To complete the proof of the claim, then, take any $a, b \in X$ with $a \succ b$. To derive a contradiction, suppose we have $b \succsim_{(x,y)} a$. Then, by definition of $\succsim_{(x,y)}$, (b, a) must belong to $x^{\uparrow,\succsim} \times y^{\downarrow,\succsim}$, and hence, $y \succsim a \succ b \succsim x$. As $\succsim$ is transitive, then, $y \succ x$, and this implies y $\mathbf{R}^{>}$ x (because $\succ$ is a subrelation of $\mathbf{R}^{>}$), but this contradicts the fact that $(x, y) \in \mathbf{R}$. $\|$

For each $(x, y) \in \mathbf{T}$, let $\succsim_{(x,y)}$ be a preorder on X that satisfies the conditions of the claim above, and put $\mathcal{P} := \{\succsim_{(x,y)} : (x, y) \in \mathbf{T}\} \cup \{\succsim\}$. Then, every element of $\mathcal{P}$ is a preorder on X that extends $\succsim$. As $\succsim$ is a subrelation of $\succsim_{(x,y)}$, and $\succsim_{(x,y)}$ is a subrelation of $\mathbf{R}$, for each $(x, y) \in \mathbf{T}$, it is also plain that $\succsim = \bigcap \mathcal{P}$ and $\bigcup \mathcal{P} \subseteq \mathbf{R}$. On the other hand, if x $\mathbf{R}$ y, then either $x \succsim y$ or $(x, y) \in \mathbf{T}$. In the former case, we obviously have $(x, y) \in \bigcup \mathcal{P}$. In the latter case, $x \succsim_{(x,y)} y$, and we again find $(x, y) \in \bigcup \mathcal{P}$. Conclusion: $\bigcup \mathcal{P} = \mathbf{R}$. Finally, as $\mathbf{R}$ is total, this finding shows that $\bigcup \mathcal{P}$ is total, and our proof is complete.

Proof of Corollary 3.4

For the "if" part of the claim, observe that the hypothesis implies $\mathbf{R}$ is a complete preorder which extends (includes) $\succsim$. Obviously, $\succsim$ is a preorder as it is the intersection of the collection $\mathcal{P}$ of preorders. This concludes that $(\succsim, \mathbf{R})$ is a transitive (weak) preference structure on X. For the "only if" part, we readily obtain the required conditions by setting $\mathcal{P} = \{\succsim, \mathbf{R}\}$.

Proof of Proposition 4.1

We will use the following preliminary result to streamline the argument.

Lemma A.1. *Let S be a nonempty set, and $\mathbf{R}$ a complete binary relation on X such that $\bigcirc(S, \mathbf{R}) \neq \varnothing$. Then, x $\mathrm{tran}(\mathbf{R}|_S)$ y for every $x, y \in \bigcirc(S, \mathbf{R})$.*

Proof. Suppose the assertion is false. By completeness of $\mathbf{R}$, then, there is a $y \in \bigcirc(S, \mathbf{R})$ such that $A := \{x \in \bigcirc(S, \mathbf{R}) : x \ \mathrm{tran}(\mathbf{R}|_S)^{>} \ y\}$ is nonempty. Then, A $\mathrm{tran}(\mathbf{R}|_S)^{>}$ z, which implies A $\mathbf{R}|_S^{>}$ z (because $\mathbf{R}$ is complete), for every $z \in \bigcirc(S, \mathbf{R}) \backslash A$. But then A is a proper subset of $\bigcirc(S, \mathbf{R})$ which is an $\mathbf{R}$-highset in S, which contradicts $\bigcirc(S, \mathbf{R})$ being the smallest such set. ■

We now turn to the proof of Proposition 4.1. If $\bigcirc(S, \mathbf{R}) \neq \varnothing$, then Lemma A.1, and the fact that $\bigcirc(S, \mathbf{R})$ is an $\mathbf{R}$-highset in S, readily entail that any one element of $\bigcirc(S, \mathbf{R})$ is a maximum element in S with respect to $\mathrm{tran}(\mathbf{R}|_S)$. In other words, nonemptiness of

$\bigcirc(S, \mathbf{R})$ entails that max(S,tran($\mathbf{R}|_S$)) is nonempty. Consequently, it is enough to prove the desired equation under the hypothesis that max(S,tran($\mathbf{R}|_S$)) $\neq \varnothing$. If x belongs to max(S,tran($\mathbf{R}|_S$)) and y is an element of S that does not, then $y \mathbf{R} x$ cannot hold, because otherwise, $y \mathbf{R} x$ tran($\mathbf{R}|_S$) S, and hence, y tran($\mathbf{R}|_S$) S, which means $y \in$ max(S,tran($\mathbf{R}|_S$)), a contradiction. As $\mathbf{R}$ is complete and max(S,tran($\mathbf{R}|_S$)) is nonempty, therefore, we conclude that max(S,tran($\mathbf{R}|_S$)) is an $\mathbf{R}$-highset in S. To derive a contradiction, suppose there is an $\mathbf{R}$-highset in S, say, B, which is a proper subset of max(S,tran($\mathbf{R}|_S$)). Take any x in max(S,tran($\mathbf{R}|_S$)) which does not belong to B, and fix an arbitrary y in B. As x tran($\mathbf{R}|_S$) y, there exist finitely many $a_1, ..., a_k \in S$ such that $x \mathbf{R} a_1 \mathbf{R} \cdots \mathbf{R} a_k \mathbf{R} y$. Then, since B is an $\mathbf{R}$-highset in S that contains y, it must also contain a_k. Continuing inductively with this argument, we see that each a_i, and in fact, x must belong to B, a contradiction. This completes our proof.

In the remaining part of this appendix, we adopt the following two conventions:

Notational Conventions: In what follows, where a preorder $\succsim$ on a nonempty set X is given (and understood from the context), we write $M(A)$ for $\mathbf{MAX}(A, \succsim)$ for any nonempty $A \subseteq X$. Further, for any nonnegative integer k, we put $[k] := \{0, ..., k\}$.

Proof of Theorem 4.3

We need the following fact for the main part of the argument.

Lemma A.2. *Let $\succsim$ be a continuous preorder on X. Then, for every $S \in \mathbf{k}(X)$ and $x \in S \backslash M(S)$, there exists a $y \in M(S)$ with $y \succ x$.*

Proof. Take any $S \in \mathbf{k}(X)$ and $x \in S \backslash M(S)$, and put $T := \{y \in S : y \succsim x\}$. By upper semicontinuity of $\succsim \cap (S \times S)$, T is a closed subset of S. Since S is compact, therefore, T is a compact set in X. Then, by means of a well-known theorem of order-theory, we have $M(T) \neq \varnothing$.[15] Pick any y in this set. Notice that any $z \in S$ with $z \succsim y$ must belong to T (by transitivity of $\succsim$). It follows that y is $\succsim$-maximal in S as well. And, obviously, $y \succsim x$. Besides, since x is not $\succsim$-maximal in S, we have $y \succ x$. ■

We now turn to the proof of Theorem 4.3. Let $(\succsim, \mathbf{R})$ be a continuous preference structure on X. Take any S in $\mathbf{k}(X)$, and note that $M(S) \neq \varnothing$ (Lemma A.2). If there is an $\mathbf{R}$-maximum element in $M(S)$, then, obviously, this element is tran($\mathbf{R}|_{M(S)}$)-maximum in $M(S)$, and hence it belongs to $\bigcirc(M(S), \mathbf{R})$. Assume, then, there is no $\mathbf{R}$-maximum in $M(S)$. This means that for every $x \in M(S)$, there is a $y \in M(S)$ with $y \mathbf{R}^{>} x$. Moreover, take an arbitrary $x \in S \backslash M(S)$, and observe that, by Lemma A.2, there exists a $z \in M(S)$ with $z \succ x$. If this x is such that $x \mathbf{R} y$ for all $y \in M(S)$, then we also have $z \mathbf{R} y$ for

[15] The earliest reference for this result seems to be Wallace (1945).

all $y \in M(S)$ by $\succsim$-transitivity of $\mathbf{R}$, which contradicts the hypothesis that there is no $\mathbf{R}$-maximum in $M(S)$. Therefore, $y \ \mathbf{R}^{>} \ x$ for some $y \in M(S)$. Conclusion: For every $x \in S$, there is a $y \in M(S)$ with $y \ \mathbf{R}^{>} \ x$. It follows that $\{y^{\downarrow\downarrow} : y \in M(S)\}$ is an open cover of S, where $y^{\downarrow\downarrow} := \{x \in S : y \ \mathbf{R}^{>} \ x\}$. Since S is compact, then, there is a finite subset T of $M(S)$ such that $\{y^{\downarrow\downarrow} : y \in T\}$ covers S. As T is finite, there is a $\text{tran}(\mathbf{R}|_{M(S)})$-maximum, say, x^*, in T. But for any $x \in M(S)$, there is a $y \in T$ with $y \ \mathbf{R}^{>} \ x$ (since $\{y^{\downarrow\downarrow} : y \in T\}$ covers S), and hence $x^* \ \text{tran}(\mathbf{R}|_{M(S)}) \ y \ \mathbf{R}^{>} \ x$, that is, $x^* \ \text{tran}(\mathbf{R}|_{M(S)}) \ x$. It follows that x^* is a $\text{tran}(\mathbf{R}|_{M(S)})$-maximum in $M(S)$. By Proposition 4.1, therefore, $x^* \in \bigcirc(M(S), \mathbf{R})$. This completes the proof of Theorem 4.3.[16]

The following result shows that, in the context of a preference structure $(\succsim, \mathbf{R})$, the asymmetric part of $\mathbf{R}$ is transitive relative to the asymmetric part of $\succsim$.

Lemma A.3. *Let $(\succsim, \mathbf{R})$ be a weak preference structure on a nonempty set X. Then,*

$$x \succsim y \ \mathbf{R}^{>} \ z \ (\textit{or } x \ \mathbf{R}^{>} \ y \succsim z) \quad \textit{implies} \quad x \ \mathbf{R}^{>} \ z$$

for every $x, y, z \in X$.

Proof. Take any $x, y, z \in X$ with $x \succsim y \ \mathbf{R}^{>} \ z$ but assume that $x \ \mathbf{R}^{>} \ z$ is false. As $\mathbf{R}$ is complete, we then have $z \ \mathbf{R} \ x$. So, $z \ \mathbf{R} \ x \succsim y$ and we find $z \ \mathbf{R} \ y$ contradicting $y \ \mathbf{R}^{>} \ z$. The analogous argument shows that $x \ \mathbf{R}^{>} \ y \succsim z$ implies $x \ \mathbf{R}^{>} \ z$ as well. ■

Proof of Proposition 4.8

Take any $S \in \mathfrak{X}$, and any $x, z \in X$ with $x \succsim z \in C(S)$. We put $T := S \cup \{x\}$; our aim is to show that $x \in C(T)$. Assume first that $x \sim z$ (where $\sim$ is the symmetric part of $\succsim$). In this case, $M(S) \cup \{x\} = M(T)$. So, in view of Proposition 4.1, $z \in C(S)$ implies that $z \ \text{tran}(\mathbf{R}|_{M(S)}) \ M(S)$ while $x \ \mathbf{R}^{=} \ z$ (because $\mathbf{R}$ contains $\succsim$). It follows that $x \ \text{tran}(\mathbf{R}|_{M(T)}) \ M(T)$, so, again by Proposition 4.1, $x \in C(T)$.

Assume now that $x \succ z$. In this case x belongs to $M(T)$, but z does not. To derive a contradiction, let us suppose that x does not belong to $C(T)$. By Proposition 4.1, then, there must exist a $y \in M(T)\backslash\{x\}$ such that

$$y \ \text{tran}(\mathbf{R}|_{M(T)})^{>} \ x. \tag{6}$$

Now, since $z \in C(S)$ and $y \in M(T)\backslash\{x\}$, and hence $y \in M(S)$, we have $z \ \text{tran}(\mathbf{R}|_{M(S)}) \ y$, so there is a positive integer k and $w_0, ..., w_k \in M(S)$ such that $z = w_0 \ \mathbf{R} \ w_1 \ \mathbf{R} \cdots \mathbf{R} \ w_k = y$. Put $\ell := \max\{i \in [k] : x \succ w_i\}$, which is well-defined because $x \succ w_0$. By (6), and because $\mathbf{R}$ contains $\succsim$, we cannot have $x \succsim w_k$, and hence $\ell \in [k-1]$. But then $w_{\ell+1}, ..., w_k \in M(T)$,

[16] This argument shows that continuity can be relaxed to the hypothesis that $x^{\uparrow,\succsim}$ and $x^{\uparrow,\mathbf{R}}$ are closed in X for every $x \in X$.

and we have $x \succ w_\ell \ \mathbf{R}\ w_{\ell+1}\ \mathbf{R} \cdots \mathbf{R}\ w_k = y$, so, by $\succsim$-transitivity of $\mathbf{R}$, $x\ \mathbf{R}\ w_{\ell+1}\ \mathbf{R} \cdots \mathbf{R}\ w_k = y$. This means $x \operatorname{tran}(\mathbf{R}|_{M(T)})\ y$, contradicting (6).

Proof of Proposition 4.9

Take any finite $S \in \mathfrak{X}$, and any $x, z \in X$ with $x\ \mathbf{R}\ z$ and $\{z\} = C(S)$. Put $T := S \cup \{x\}$; we wish to show that $x \in C(T)$. Suppose first that x is not $\succsim$-maximal in T. Then $y \succ x$ for some $y \in T$. Since T is finite and $\succsim$ is transitive, it is without loss of generality to assume that $y \in M(T)$. Since $y \succ x\ \mathbf{R}\ z$, we get $y\ \mathbf{R}\ z$ by $\succsim$-transitivity of $\mathbf{R}$. As $\{z\}$ is the top-cycle in $M(S)$, and $y \in M(S)$, we must then have $y = z$. But this means $z \succ x$, and hence $z\ \mathbf{R}^{>}\ x$, contradiction. Conclusion: $x \in M(T)$.

Now, since $x\ \mathbf{R}\ z$, and $\mathbf{R}$ extends $\succsim$, we do not have $z \succ x$. On the other hand, by Proposition 4.8, $x \succsim z$ implies $x \in C(T)$. It remains to consider the case where $(x, z) \in \operatorname{Inc}(\succsim)$. In this case, $z \in M(T)$. Moreover, as $\{z\}$ is the top-cycle in $M(S)$, we have $z\ \mathbf{R}^{>}\ y$ for every $y \in M(S)\backslash\{z\}$. But then, $x\ \mathbf{R}\ z\ \mathbf{R}^{>}\ y$, and hence $x \operatorname{tran}(\mathbf{R}|_{M(T)})\ y$, for every $y \in M(T)\backslash\{x, z\}$. By Proposition 4.1, then, $x \in C(T)$, and we are done.

Proof of Theorem 5.2

We begin with proving a preliminary result that will be needed in the main body of the proof. This lemma is stated in the setting of Theorem 5.2.

Lemma A.4. *For any $S \in \mathfrak{X}$, $\succsim \in \mathbb{P}(C)$, and any $(x, y) \in S \times X$ with $x \succsim y$,*

$$x \succsim y \quad \textit{implies} \quad C(S \cup \{y\}) \cap S = C(S).$$

Proof. If y is not $\succsim$-maximal in $S \cup \{y\}$, then $M(S \cup \{y\}) = M(S)$, and the claim follows readily from Proposition 4.1. We thus assume that $y \in M(S \cup \{y\})$. In turn, since $x \succsim y$, this implies that $x \in \mathbf{MAX}(S \cup \{y\})$ and $x \sim y$. Consequently,

$$\{x, y\} \subseteq M(S \cup \{y\}) = M(S) \cup \{y\}. \tag{7}$$

Besides, for any $a, b \in M(S)$, we have

$$a \operatorname{tran}(\mathbf{R}|_{M(S)})\ b \quad \text{iff} \quad a \operatorname{tran}(\mathbf{R}|_{M(S)\cup\{y\}})\ b, \tag{8}$$

where we denote $\mathbf{R}_C$ by $\mathbf{R}$ to simplify the notation. (The "only if" part of (8) is trivial. Its "if" part follows from the fact that $z\ \mathbf{R}\ y$ implies $z\ \mathbf{R}\ x$, and $y\ \mathbf{R}\ z$ implies $x\ \mathbf{R}\ z$, for any $z \in S$ (because $x \sim y$, and $\mathbf{R}$ is $\succsim$-transitive).) Now, there are three cases to consider.

Case 1. $C(S) = \varnothing$. In this case, by Proposition 4.1, there is no $\operatorname{tran}(\mathbf{R}|_{M(S)})$-maximum in $M(S)$. It then follows from (8) that there is no $\operatorname{tran}(\mathbf{R}|_{M(S)\cup\{y\}})$-maximum in $M(S)\cup\{y\}$. Since $M(S \cup \{y\}) = M(S) \cup \{y\}$, then, Proposition 4.1 entails $C(S \cup \{y\}) \cap S = \varnothing$.

Case 2. $C(S) \neq \varnothing$ and $x \in S \backslash C(S)$. In this case, we have $C(S)$ $\mathbf{R}^{>}$ $x \sim y$. So, by Lemma A.3, $C(S)$ $\mathbf{R}^{>}$ y, that is, $C(S)$ is an $\mathbf{R}$-highset in $M(S) \cup \{y\}$. As $C(S)$ is obviously an $\mathbf{R}$-cycle and (7) holds, we conclude, by Corollary 4.2, that $C(S)$ is the top-cycle in $M(S \cup \{y\})$ with respect to $\mathbf{R}$, that is, $C(S) = C(S \cup \{y\})$.

Case 3. $x \in C(S)$. Again, obviously, $C(S)$ is an $\mathbf{R}$-cycle. Since $x \sim y$ and $\mathbf{R}$ extends $\succsim$, therefore, $C(S) \cup \{y\}$ is an $\mathbf{R}$-cycle as well. Moreover, $y \sim x$ $\mathbf{R}^{>}$ $M(S) \backslash C(S)$ and hence, y $\mathbf{R}^{>}$ $M(S) \backslash C(S)$ by Lemma A.3. It follows that $C(S) \cup \{y\}$ is an $\mathbf{R}$-highset in $M(S) \cup \{y\}$. In view of (7) and Corollary 4.2, then, $C(S) \cup \{y\}$ is the top-cycle in $M(S \cup \{y\})$ with respect to $\mathbf{R}$, and hence, $C(S) \cup \{y\} = C(S \cup \{y\})$. ■

We now turn to the proof of Theorem 5.2 in which $\mathcal{P}$ stands for an arbitrarily fixed nonempty subset of $\mathbb{P}(C)$. We define $\trianglerighteq_{\mathcal{P}} := \text{tran}(\bigcup \mathcal{P})$, and write $\triangleright_{\mathcal{P}}$ for the asymmetric part of this preorder. (We wish to show that $\trianglerighteq_{\mathcal{P}}$ is the supremum of $\mathcal{P}$ in $\mathbb{P}(C)$ relative to the partial order $\sqsupseteq$.) We organize our main argument in terms of several claims.

Claim 1. $\trianglerighteq_{\mathcal{P}}$ extends any member of $\mathcal{P}$.

Proof of Claim 1. Take any $\succsim$ in $\mathcal{P}$. Obviously, $\trianglerighteq_{\mathcal{P}}$ contains $\succsim$. Next, take any $x, y \in X$ with $x \succ y$. To derive a contradiction, suppose $x \triangleright_{\mathcal{P}} y$ does not hold. Since $x \trianglerighteq_{\mathcal{P}} y$, this means that $y \trianglerighteq_{\mathcal{P}} x$ holds as well. By definition of $\trianglerighteq_{\mathcal{P}}$, then, there exist a $k \in \mathbb{N}$, $\succsim_1, ..., \succsim_k \in \mathbb{P}(C)$ and $z_0, ..., z_k \in X$ such that $x \succ y = z_0 \succsim_1 z_1 \succsim_2 \cdots \succsim_k z_k = x$. Put $S := \{z_0, ..., z_k\}$. Since $C(S) \neq \varnothing$, there is an $i \in [k]$ such that $z_i \in C(S)$. If $i > 0$, Proposition 4.8 entails that $z_{i-1} \in C(S)$, and continuing inductively, we find that $y = z_0 \in C(S)$. So, in all contingencies, we have $y \in C(S)$. But this is impossible, because y is not $\succsim$-maximal in S, and $(\succsim, \mathbf{R}_C)$ rationalizes C. □

Claim 2. $(\trianglerighteq_{\mathcal{P}}, \mathbf{R}_C)$ is a preference structure on X.

Proof of Claim 2. Let us first show that $\mathbf{R}_C$ extends $\trianglerighteq_{\mathcal{P}}$. Take any $x, y \in X$ with $x \trianglerighteq_{\mathcal{P}} y$. Then, there exist a $k \in \mathbb{N}$, $\succsim_1, ..., \succsim_k \in \mathbb{P}(C)$ and $z_0, ..., z_k \in X$ such that

$$x = z_0 \succsim_1 z_1 \succsim_2 \cdots \succsim_k z_k = y. \tag{9}$$

If $k = 1$, then $x \succsim_1 y$, and hence x $\mathbf{R}_C$ y (because $\mathbf{R}_C$ is a superrelation of $\succsim_1$). Suppose $k \geq 2$. Then, $z_{k-2} \succsim_{k-1} z_{k-1} \succsim_k z_k$, and hence $z_{k-2} \succsim_{k-1} z_{k-1}$ $\mathbf{R}_C$ z_k, so by $\succsim_{k-1}$-transitivity of $\mathbf{R}_C$, we find

$$x = z_0 \succsim_1 z_1 \succsim_2 \cdots \succsim_{k-2} z_{k-2} \mathbf{R}_C\ z_k = y.$$

If $k = 2$, we are done. Otherwise, we continue this way inductively to obtain x $\mathbf{R}_C$ y in $k-1$ steps. Conclusion: x $\mathbf{R}_C$ y. Now assume that we in fact had $x \triangleright_{\mathcal{P}} y$ (which implies that at least one of the orderings in (9) holds strictly). Let us show that x $\mathbf{R}_C^{=}$ y could not hold in this case. Indeed, to derive a contradiction, suppose x $\mathbf{R}_C^{=}$ y so that $y \in C\{x, y\}$. Then, since

$x \succsim_1 z_1$, Lemma A.4 entails that $C\{x,y\} \subseteq C\{x,y,z_1\}$, so $y \in C\{x,y,z_1\}$. If $k \geq 2$, then, since $z_1 \succsim_1 z_2$, Lemma A.4 entails that $y \in C\{x,y,z_1\} \subseteq C\{x,y,z_1,z_2\}$. Continuing this way inductively, we find that $y \in C\{x,y,z_1,...,z_{k-1}\}$. But then, since $z_{k-1} \succsim_k y$ and $(\succsim_k, \mathbf{R}_C)$ rationalizes C, Proposition 4.8 yields $z_{k-1} \in C\{x,y,z_1,...,z_{k-1}\}$. Continuing this way inductively, therefore, we find $\{x,y,z_1,...,z_{k-1}\} = C\{x,y,z_1,...,z_{k-1}\}$, but this contradicts the fact that at least one of the orderings in (9) holds strictly. Conclusion: $x \rhd_{\mathcal{P}} y$ implies $x\ \mathbf{R}_C^{>}\ y$.

It remains to show that $\mathbf{R}_C$ is $\trianglerighteq_{\mathcal{P}}$-transitive. To this end, take any $x, y, z \in X$ with $x \trianglerighteq_{\mathcal{P}} y\ \mathbf{R}_C\ z$. Then, there are $k \in \mathbb{N}$, $\succsim_1, ..., \succsim_k \in \mathbb{P}(C)$ and $w_0, ..., w_k \in X$ such that $x = w_0 \succsim_1 w_1 \succsim_2 \cdots \succsim_k w_k = y\ \mathbf{R}_C\ z$. So, repeating the induction argument we gave in the previous paragraph, we find $x\ \mathbf{R}_C\ z$. That $x\ \mathbf{R}_C\ y \trianglerighteq_{\mathcal{P}} z$ implies $x\ \mathbf{R}_C\ z$ is similarly proved. $\square$

Claim 3. For any $S \in \mathfrak{X}$ and $\succsim \in \mathcal{P}$,

$$C(S) \subseteq \mathbf{MAX}(S, \trianglerighteq_{\mathcal{P}}) \subseteq \mathbf{MAX}(S, \succsim).$$

Proof of Claim 3. The second containment is an immediate consequence of Claim 1. To establish the first containment, take any x in $C(S)$, and suppose that x is not $\trianglerighteq_{\mathcal{P}}$-maximal in S. Then, there is a $y \in S$ with $y \rhd_{\mathcal{P}} x$, and hence, $y = z_0 \succsim_1 z_1 \succsim_2 \cdots \succsim_k z_k = x$ for some $k \in \mathbb{N}$, $\succsim_1, ..., \succsim_k \in \mathbb{P}(C)$ and $z_0, ..., z_k \in X$, with at least one of these ordering holding strictly. (We have $k > 1$, for otherwise $y \succ_1 x$, that is, x is not $\succsim_1$-maximal in S, contradicting $x \in C(S)$.) Since $y \in S$ and $y \succsim_1 z_1$, Lemma A.4 tells us that $C(S \cup \{z_1\}) \cap S = C(S)$. But then, since $z_1 \in S \cup \{z_1\}$ and $z_1 \succsim_2 z_2$, applying Lemma A.4 again yields

$$C(S \cup \{z_1, z_2\}) \cap (S \cup \{z_1\}) = C(S \cup \{z_1\}).$$

Intersecting both sides of this equation with S, and using the previous equation, then, we find

$$C(S \cup \{z_1, z_2\}) \cap S = C(S).$$

In fact, proceeding this way inductively, we may conclude that

$$C(S \cup \{z_1, ..., z_k\}) \cap S = C(S). \tag{10}$$

In particular, x belongs to $C(S \cup \{z_1, ..., z_k\})$. But then, by Proposition 4.8, z_{k-1} belongs to $C(S \cup \{z_1, ..., z_k\})$ as well. In fact, applying Proposition 4.8 this way inductively, we find that $z_0, ..., z_k \in C(S \cup \{z_1, ..., z_k\})$. But this is impossible, for $z_{i-1} \succ_i z_i$ holds for at least one $i \in \{1, ..., k\}$, so, being not $\succsim_i$-maximal in $S \cup \{z_1, ..., z_k\}$, z_i cannot belong to $C(S \cup \{z_1, ..., z_k\})$. $\square$

Claim 4. $(\trianglerighteq_{\mathcal{P}}, \mathbf{R}_C)$ rationalizes C.

Proof of Claim 4. Take any $S \in \mathfrak{X}$ and note that $C(S)$ is an $\mathbf{R}_C$-cycle in $C(S)$. So, since, by Claim 3, $C(S) \subseteq \mathbf{MAX}(S, \trianglerighteq_{\mathcal{P}})$, it is plain that $C(S)$ is an $\mathbf{R}_C$-cycle in $\mathbf{MAX}(S, \trianglerighteq_{\mathcal{P}})$. Now pick any $\succsim \in \mathbb{P}(C)$. Then, $C(S)$ is an $\mathbf{R}_C$-highset in $\mathbf{MAX}(S, \succsim)$. By Claim 3, therefore, $C(S)$ is an $\mathbf{R}_C$-highset in $\mathbf{MAX}(S, \trianglerighteq_{\mathcal{P}})$ as well. This means that $C(S)$ is the top-cycle in $\mathbf{MAX}(S, \trianglerighteq_{\mathcal{P}})$ with respect to $\mathbf{R}_C$, as we claimed. □

Claims 2 and 4 jointly imply that $\trianglerighteq_{\mathcal{P}}$ belongs to $\mathbb{P}(C)$. It then follows from Claim 1 that $\trianglerighteq_{\mathcal{P}}$ is the supremum of $\mathcal{P}$ in $\mathbb{P}(C)$ relative to $\sqsupseteq$. In view of the arbitrary choice of $\mathcal{P}$ above, we conclude that $\mathbb{P}(C)$ is a complete $\vee$-semilattice relative to this partial order.

Proof of Theorem 5.3

Throughout the proof, we will denote $\mathbf{R}_C$ by $\mathbf{R}$ to simplify the notation. (That is, for any x and y in X, we have $\{x\} = C\{x, y\}$ iff $x \ \mathbf{R}^{>} \ y$, and $\{x, y\} = C\{x, y\}$ iff $x \ \mathbf{R}^{=} \ y$.) Consequently, $\succ_C \subseteq \mathbf{R}^{>}$ and $\sim_C \subseteq \mathbf{R}^{=}$, that is, that $\mathbf{R}$ extends $\succsim_C$. We will use these facts below as a matter of routine. Also, when $\succsim \in \mathbb{P}(C)$, we again write $M(S) = \mathbf{MAX}(S, \succsim)$ for any $S \in \mathfrak{X}$. The following lemmata are stated in the setting of Theorem 5.3.

Lemma A.5. *For any finite $S \in \mathfrak{X}$,*

$$\{x\} = C(S) \quad \text{implies} \quad \{x\} = C\{x, y\} \text{ for every } y \in S.$$

Proof. Take any finite $S \in \mathfrak{X}$ with $\{x\} = C(S)$. If $S = \{x\}$, there is nothing to prove, so assume otherwise, and pick any $y \in S \backslash \{x\}$. Take any $\succsim \in \mathbb{P}(C)$, and denote $\mathbf{R}_C$ by $\mathbf{R}$ to simplify the notation. If $y \in M(S)$, then since $\{x\}$ is the top-cycle in $M(S)$ with respect to $\mathbf{R}$, we have $x \ \mathbf{R}^{>} \ y$. If y is not $\succsim$-maximal in S, then $z \succ y$ holds for some $z \in S$. As S is finite, we may assume that $z \in M(S)$. If $z = x$, then, obviously, $\{x\} = C\{x, y\}$, so assume that $z \neq x$. Then, since $\{x\} = \bigcirc(M(S), \mathbf{R})$, we have $x \ \mathbf{R}^{>} \ z \succ y$, so, by Lemma A.3, we again find $x \ \mathbf{R}^{>} \ y$. Since $\{x\} = C\{x, y\}$ iff $x \ \mathbf{R}^{>} \ y$, we are done. ■

Lemma A.6. *For any $S \in \mathfrak{X}$ and $x \in S$,*

$$x \succ_C y \quad \text{implies} \quad C(S \cup \{y\}) = C(S).$$

Proof. Take any $S \in \mathfrak{X}$. Let x and y be two elements of X with $x \succ_C y$. Then, $x \ \mathbf{R}^{>} \ y$, so $y \succsim x$ cannot hold (because $\mathbf{R}$ extends $\succsim$). Besides, if there is a $z \in S$ with $z \succ y$, then $C(S \cup \{y\}) \subseteq S$, so, by Lemma A.4, $C(S \cup \{y\}) = C(S)$, and we are done. It remains to consider the case where $(x, y) \in \text{Inc}(\succsim)$ and $y \in M(S \cup \{y\})$.

Since $x \succ_C y$ implies that y does not belong to $C(S \cup \{y\})$, we have

$$C(S \cup \{y\}) \subseteq M(S \cup \{y\}) \cap S \subseteq M(S).$$

Now, we claim that $C(S \cup \{y\})$ is an $\mathbf{R}$-highset in $M(S)$ To see this, take any $\succsim$-maximal z in S that does not belong to $C(S \cup \{y\})$. If $z \in M(S \cup \{y\})$, then we clearly have $C(S \cup \{y\})$ $\mathbf{R}^{>}$ z (because $C(S \cup \{y\})$ is an $\mathbf{R}$-highset in $M(S \cup \{y\})$. If $z \notin M(S \cup \{y\})$, then, since $y \in M(S \cup \{y\}) \backslash C(S \cup \{y\})$, we have $C(S \cup \{y\})$ $\mathbf{R}^{>}$ $y \succ z$, which implies $C(S \cup \{y\})$ $\mathbf{R}^{>}$ z by Lemma A.3. So, $C(S \cup \{y\})$ is an $\mathbf{R}$-highset in $M(S)$. As $C(S \cup \{y\})$ is obviously an $\mathbf{R}$-cycle, $C(S \cup \{y\}) = C(S)$ by Corollary 4.2. ■

We now turn to the proof of Theorem 5.3.

Claim 1. $\mathbf{R}$ is $\succsim_C$-transitive.

Proof of Claim 1. Let us first show that $\mathbf{R}$ is $\succ_C$-transitive. Take any $x, y, z \in X$ with x $\mathbf{R}$ $y \succ_C z$. If x $\mathbf{R}$ z is false, then z $\mathbf{R}^{>}$ x (because $\mathbf{R}$ is complete). Take any $\succsim$ in $\mathbb{P}(C)$. Clearly, $y \succ x$ cannot hold (because $\succsim \subseteq \mathbf{R}$). If, on the other hand, $x \succ y$, then $\{x\} = C\{x, y, z\}$, and by Lemma A.5, this implies x $\mathbf{R}^{>}$ z, a contradiction. Thus: $(x, y) \in \operatorname{Inc}(\succ)$. Similarly, $x \succ z$ cannot hold (because $\succsim \subseteq \mathbf{R}$), and if $z \succ x$, then $\{y\} = C\{x, y, z\}$, and by Lemma A.5, this implies y $\mathbf{R}^{>}$ x, a contradiction. Thus: $(x, z) \in \operatorname{Inc}(\succ)$. Finally, note that $z \succ y$ cannot hold (because $\succ \subseteq \succ_C$), and if $y \succ z$, then Lemma A.3 implies y $\mathbf{R}^{>}$ x, a contradiction. Thus: $(y, z) \in \operatorname{Inc}(\succ)$. Conclusion: $\mathbf{MAX}(\{x, y, z\}, \succsim) = \{x, y, z\}$. Then, by Proposition 4.1, $\{x, y, z\} = C\{x, y, z\}$, but this contradicts $y \succ_C z$. Thus: $\mathbf{R} \circ \succ_C$ $\subseteq \mathbf{R}$. One can similarly prove that $\succ_C \circ \mathbf{R} \subseteq \mathbf{R}$.

We next show that $\mathbf{R}$ is $\sim_C$-transitive. Take any $x, y, z \in X$ with $x \,\mathbf{R}\, y \sim_C z$. The second part of this statement entails that $x \in C(\{x\} \cup \{y\})$ iff $x \in C(\{x\} \cup \{z\})$. But $x \in C\{x, y\}$ (because x $\mathbf{R}$ y), so we find that $x \in C\{x, z\}$, that is, x $\mathbf{R}$ z. We thus conclude that $\mathbf{R} \circ \sim_C \subseteq \mathbf{R}$. One can similarly prove that $\sim_C \circ \mathbf{R} \subseteq \mathbf{R}$. □

Claim 2. $\succsim_C$ is transitive.

Proof of Claim 2. Observe that it is sufficient to verify (i) $\sim_C$ is transitive, (ii) $\succ_C$ is transitive, and (iii) $\succ_C$ is $\sim_C$-transitive. The construction of $\sim_C$ readily implies (i). For (ii), suppose that $x \succ_C y \succ_C z$ but $x \succ_C z$ does not hold for some $x, y, z \in X$. Then, there exists an $S \in \mathfrak{X}$ with $x \in S$ and $z \in C(S)$. As $x \succ_C y$, Lemma A.6 implies that $z \in C(S) = C(S \cup \{y\})$. But this is a contradiction since $y \succ_C z$. Thus, $x \succ_C y \succ_C z$ must imply $x \succ_C z$. For (iii), take any $x, y, z \in X$ such that $x \succ_C y \sim_C z$. Let S be an arbitrary element of $\mathfrak{X}$ with $x \in S$. (We wish to show that z does not belong to $C(S)$.) If z does not belong to S, there is nothing to prove, so suppose $z \in S$. As $x \succ_C y$ and $x \in S$, we have $y \notin C(S \cup \{y\})$. Thus, since $y \sim_C z$, we have $z \notin C(S \cup \{z\}) = C(S)$. An analogous argument shows that $x \sim_C y \succ_C z$ implies $x \succ_C z$ as well. □

Claim 3. For any $\succsim \in \mathbb{P}(C)$, $\succsim_C$ extends $\succsim$.

Proof of Claim 3. Let $\succsim \in \mathbb{P}(C)$. If $x \succ y$ for some $x, y \in X$, then $y \notin M(S)$ and thus $y \notin C(S)$ for all $S \in \mathfrak{X}$ with $x \in S$. So, $\succ \subseteq \succ_C$. In the rest of the proof, we prove that

$\sim \subseteq \sim_C$. Take any $x, y \in X$ with $x \sim y$. Let S be an arbitrarily fixed element of $\mathfrak{X}$, and put $T_x := M(S \cup \{x\})$ and $T_y := M(S \cup \{y\})$. Since $\succsim$ is a preorder, it is readily checked that $x \sim y$ implies $T := T_x \cap S = T_y \cap S$.

Now assume $x \in C(S\cup\{x\})$. Then, $x \in M(S\cup\{x\})$, and hence, $y \in M(S\cup\{y\})$ (because $\succsim$ is a preorder and $x \sim y$), that is, $y \in T_y$. Moreover, by Proposition 4.1, x $\text{tran}(\mathbf{R}|_{T_x})$ T_x. So, if $z \in T \subseteq T_x$, there is a positive integer k such that $x \mathbf{R} w_0 \mathbf{R} \cdots \mathbf{R} w_k = z$ for some $w_0, ..., w_k \in T_x$. Here, we can in fact assume that $w_0, ..., w_k \in T$ without loss of generality. (For, otherwise, $w_i = x$ for some $i \in [k]$. Then, set $l := \max\{i \in [k] : w_i = x\}$, and we have x $\mathbf{R} w_{l+1} \mathbf{R} \cdots \mathbf{R} w_k = z$ with $w_i \in T$ for all $i = l+1, \ldots, k$.) Since $y \sim x$ and $\mathbf{R}$ is $\succsim$-transitive, this implies $y \mathbf{R} w_0 \mathbf{R} \cdots \mathbf{R} w_k = z$. Thus, y $\text{tran}(\mathbf{R}|_{T_y})$ T. If $z \in T_y \backslash T$, then $z = y$, and we obviously have y $\text{tran}(\mathbf{R}|_{T_y})$ z. Conclusion: y $\text{tran}(\mathbf{R}|_{T_y})$ T_y. By Proposition 4.1, this yields $y \in C(S \cup \{y\})$, as we sought. By symmetry, therefore, we conclude: $x \in C(S \cup \{x\})$ iff $y \in C(S \cup \{y\})$.

Next, take any z in S with $z \in C(S \cup \{x\})$. Then, $z \in M(S \cup \{x\})$, and hence, $z \in M(S \cup \{y\})$ (because $\succsim$ is a preorder and $x \sim y$), that is, $z \in T_y$. Moreover, by Proposition 4.1, z $\text{tran}(\mathbf{R}|_{T_x})$ T_x. Now, take any $w \in T_y$. Then, we can show that there is a positive integer k with

$$z \mathbf{R} w_0 \mathbf{R} w_1 \mathbf{R} \cdots \mathbf{R} w_k \mathbf{R} w \quad \text{for some } w_0, \ldots, w_k \in T_x. \tag{11}$$

(Indeed, if $w \in T \subseteq T_x$, then z $\text{tran}(\mathbf{R}|_{T_x})$ w, and (11) follows at once. If $w \in T_y \setminus T$, then $y = w \in T_y$, which implies $x \in T_x$ and hence z $\text{tran}(\mathbf{R}|_{T_x})$ $x \sim y$. So, there is a positive interger k such that $z \mathbf{R} w_0 \mathbf{R} w_1 \mathbf{R} \cdots \mathbf{R} w_k \mathbf{R} x \sim y$ for some $w_0, \ldots, w_k \in T_x$. This again implies (11) by $\succsim$-transitivity of $\mathbf{R}$.) For the sequence $w_0, ..., w_k$ in (11), define

$$w_i' := \begin{cases} w_i, & \text{if } w_i \neq x \\ y, & \text{if } w_i = x, \end{cases}$$

for each $i \in [k]$, and note that $z \mathbf{R} w_1' \mathbf{R} \cdots \mathbf{R} w_k' \mathbf{R} w$ by $\succsim$-transitivity of $\mathbf{R}$. Since $w_i' \in T_y$ for each $i \in [k]$, this shows that z $\text{tran}(\mathbf{R}|_{T_y})$ w. It then follows from the arbitrary choice of w that z $\text{tran}(\mathbf{R}|_{T_y})$ T_y, that is, $z \in C(S \cup \{y\})$, as we sought. By symmetry, therefore, we conclude: $z \in C(S \cup \{x\})$ iff $z \in C(S \cup \{y\})$ for every $z \in S$. In view of the arbitrariness of S, this establishes that $\sim \subseteq \sim_C$. $\square$

In view of Theorem 5.2, we have $\bigvee \mathbb{P}(C) \in \mathbb{P}(C)$. So, by Claim 3, it follows that $\bigvee \mathbb{P}(C) \subseteq \succsim_C$. As $\mathbf{R}$ extends $\succsim_C$, Claim 1 and Claim 2 imply that $(\succsim_C, \mathbf{R})$ is a preference structure on X. Since $\bigvee \mathbb{P}(C)$ is the largest preorder in $\mathbb{P}(C)$, if $(\succsim_C, \mathbf{R})$ rationalizes C, then $\succsim_C \subseteq \bigvee \mathbb{P}(C)$. The next claim estabilishes this step, hence completing the proof of Theorem 5.3.

Claim 4. $(\succsim_C, \mathbf{R})$ rationalizes C.

Proof of Claim 4. Take any S in $\mathfrak{X}$ and $\succsim \in \mathbb{P}(C)$. If $x \in C(S)$, then $y \succ_C x$ holds for no $y \in S$ by definition of $\succ_C$, implying that $x \in \mathbf{MAX}(S, \succsim_C)$. So, $C(S) \subseteq \mathbf{MAX}(S, \succsim_C)$. In addition, we have $\mathbf{MAX}(S, \succsim_C) \subseteq M(S)$ as $\succsim_C$ extends $\succsim$ by Claim 3. These observations readily imply that $C(S)$ is an $\mathbf{R}$-highset in $\mathbf{MAX}(S, \succsim_C)$. (Indeed, if $x \in C(S)$ and $y \in \mathbf{MAX}(S, \succsim_C) \backslash C(S)$, then $y \in M(C) \backslash C(S)$ and thus $x \, \mathbf{R}^{>} y$.) As $C(S)$ is obviously an $\mathbf{R}$-cycle, we conclude that $C(S) = \bigcirc(\mathbf{MAX}(S, \succsim_C), \mathbf{R})$ by Corollary 4.2. $\square$

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